PHD ANALYSIS QUALIFYING EXAM, JANUARY 2026

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Do six of eight.

(1) Do two of three.

(i) Give an example of a Lebesgue measurable set $E \subseteq [0,1]$ that is closed, has positive measure, but contains no non-trivial interval or explain why no such example exists.

(ii) Is the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{1}{1+x^2}$ the Hardy-Littlewood maximal function of an $\mathcal{L}^1(\mathbb{R})$ function f?

(iii) Does there exist a measure space $(X, \mathscr{M}, \mu)$ and a nested decreasing sequence $(E_n)_{n=1}^{\infty}$ of measurable sets such that

$$\lim_{n\to\infty} \mu(E_n) \neq \mu(\cap_{n=1}^{\infty} E_n)?$$

(2) Suppose $f : [0,1] \times [0,1] \to \mathbb{R}$. Show, if

(a) for each $x \in [0,1]$ the function $g_x(y) = f(x,y)$ is continuous;
(b) for each $y \in [0,1]$ the function $h_y(x) = f(x,y)$ is continuous; and
(c) f is bounded,

then $F : [0,1] \to \mathbb{R}$ defined by

$$F(x) = \int_{[0,1]} f(x,y)\, d\lambda(y)$$

is continuous. Here λ denotes Lebesgue measure on $\mathbb{R}$.

(3) Let $(X, \mathscr{M})$ denote a measurable space and let $P(X)$ denote the power set of X. Prove, if $\mathscr{M} \neq P(X)$ and $\{x\} \in \mathscr{M}$ for each $x \in X$, then there exists an $A \notin \mathscr{M} \otimes \mathscr{M}$ such that $[A]_x$, $[A]^y \in \mathscr{M}$ for all $x \in X$. Suggestion: First show $\delta : X \to X \times X$ defined by $\delta(x) = (x,x)$ is measurable from $(X, \mathscr{M})$ to $(X \times X, \mathscr{M} \otimes \mathscr{M})$.

(4) Suppose $(X, \mathscr{M}, \mu)$ is a measure space and $f, g : X \to [0, \infty)$ are measurable. Show

(a) $\mu_f : \mathscr{M} \to [0, \infty]$ defined by $\mu_f(A) = \int \chi_A f\, d\mu = \int_A f\, d\mu$ is a measure; and

(b) $\int g\, d\mu_f = \int gf\, d\mu$.

(5) Do two of three. Short answer: you do not need to give a detailed proof, just a brief explanation.

(a) If $a_1, \ldots, a_n$ are nonnegative real numbers, must it be the case that

$$\left(\sum_{j=1}^{n} a_j\right)^6 \leq n^5 \sum_{j=1}^{n} a_j^6 \, ?$$

(b) Explain why, given $x \in [0,1]$, the mapping $E_x : C([0,1]) \to \mathbb{C}$ defined by $E_x(f) = f(x)$ is a bounded linear functional. Is the dual of the Banach space $C([0,1])$ separable?

(c) Give an example where the Radon-Nikodym Theorem fails without the sigma-finite hypothesis.

(6) Let $\mathbb{C}[x]$ denote the complex vector space of polynomials in x. Does there exist a norm $\|\cdot\|$ on $\mathbb{C}[x]$ such that $(\mathbb{C}[x], \|\cdot\|)$ is a Banach space. Provide a proof for your answer.

(7) Suppose ϕ, ψ are bounded linear functionals on a Hilbert space V. Show, if $\|\phi+\psi\| = \|\phi\| + \|\psi\|$, then one of ϕ, ψ is a multiple of the other. Is the result still true if V is only assumed to be a Banach space?

(8) Show if $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, then the convolution $f * g$ is continuous.

Let $E \subseteq [0,1]$ be a Lebesgue measurable set of positive measure and let

$$\text{-}E = \{\text{-}x : x \in E\}.$$

By considering the function $h = \chi_{\text{-}E} * \chi_E$ or otherwise, show

$$E - E = \{x - y : x, y \in E\}$$

contains an open interval.