PhD Analysis Examination
August 2024

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. For any two of the following, either give an example or explain why no example exists.
   a) a sequence of Lebesgue measurable functions on $\mathbb{R}$ which converges in measure but not Lebesgue a.e.
   b) a closed set $E \subset [0,1]$ with positive Lebesgue measure, but which contains no open intervals
   c) a decreasing sequence of nonnegative measurable functions on $\mathbb{R}$ such that $\lim \int f_n \neq \int \lim f_n$
2. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.
3. a) State the Lebesgue-Radon-Nikodym theorem. b) State and prove a version of the chain rule for Radon-Nikodym derivatives.
4. Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a Lebesgue measurable function with the property that $fg \in L^1(\mathbb{R})$ for every $g \in L^2(\mathbb{R})$. Must f belong to $L^2(\mathbb{R})$? Prove, or give a counterexample.
5. a) Show that for every integer $n \geq 1$, there exists a function $f_n \in L^2[0,1]$ such that for every polynomial p of degree at most n,
   $$p(0) = \int_0^1 p(x) f_n(x)\, dx.$$
   b) Prove that there is *no* L^2 function f such that the above holds for *all* polynomials p (independent of the degree n).
6. State the Baire category theorem. Prove that if $\mathcal{X}$ is an infinite-dimensional Banach space, then $\mathcal{X}$ does not have a countable basis.
7. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ a normed vector space. Let (T_n) be a sequence of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. Supposed that $\lim_n T_n x$ exists for each $x \in \mathcal{X}$. Prove that the linear transformation $Tx := \lim_n T_n x$ is bounded from $\mathcal{X}$ to $\mathcal{Y}$.
8. For each of the following statements, determine if it is true or false, and sketch a proof of your claim.

a) If ν is a signed measure and E, F are measurable sets with $\nu(E) \geq 0$ and $\nu(F) \geq 0$, then $\nu(E \cup F) \geq 0$.

b) If $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are measurable spaces and $E \subset X$, $F \subset Y$ are sets such that $E \times F$ is $\mathscr{M} \otimes \mathscr{N}$-measurable, then $E \in \mathscr{M}$ and $F \in \mathscr{N}$.

c) $L^1(\mathbb{R})$ is reflexive.