Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and circle the numbers of the seven problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

Note. Below *ring* means associative ring with identity, and *module* means unital module unless otherwise specified.

1. (10 points) Let $q > 1$ be a power of a prime p. Prove that there exists some field F with $|F| = q$.

2. (10 points) Determine the Galois group of $x^4 - 5$ over $\mathbf{Q}$, over $\mathbf{Q}(\sqrt{5})$, and over $\mathbf{Q}(\sqrt{5}i)$. Justify your answers.

3. (10 points) Let R be an integral domain, let D be a divisible R-module, and let T be a torsion R-module. Prove that $D \otimes_R T = \{0\}$.

4. (10 points) Prove that the group G which is described below using generators and relations is not solvable.
$$G = \langle x, y \mid x^6 = y^2 = 1 \rangle$$

5. Let R be a ring with identity.
   (a) (3 points) Define what it means for a left R-module to be projective.
   (b) (4 points) Show that a free R-module is projective.
   (c) (3 points) Give an example of a projective R-module that is not free.

6. (10 points) Let K be a field. Show that any vector space over K has a basis (do not assume the vector space has finite dimension).

7. (10 points) Prove: If a Dedekind domain has only a finite number of nonzero prime ideals then it is a principal ideal domain. (Hint: Prove first that each prime ideal is principal, use the Chinese Remainder Theorem).

8. Let R be a integral domain.
   (a) (3 points) Define what it means for a fractional ideal of R to be invertible.
   (b) (7 points) Show that an invertible ideal is a projective R-module.

9. (10 points) Suppose $R \subseteq S$ are integral domains and let $s \in S$. Prove that the following are equivalent:
   1. s is integral over R;
   2. $R[s]$ is finitely generated as an R-module;
   3. There exists some ring T such that $R \subseteq T \subseteq S$, and $s \in T$, and T is finitely generated as an R-module.

10. (10 points) Suppose R is a ring with identity (not necessarily commutative), M is a right R-module and
$$N' \to N \to N'' \to 0$$
is an exact sequence of left R-modules. Show that
$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$
is an exact sequence of abelian groups.

11. (a) (5 points) State the Jacobson density theorem.
    (b) (5 points) Use it to show that if K is a field and A is a simple K-algebra of finite dimension then there is a division K-algebra D and a finite dimensional vector space V over D such that $A \simeq \mathrm{End}_D(V)$.