PhD Algebra Exam **August, 2024**

Answer seven problems, and on the list below circle the problems you wish to have graded. If more than seven problems are answered only the first seven will be graded. Write your answers clearly in complete English sentences.

Results from lectures or textbooks may be used without proof (within reason – don't state results equivalent to the problem), but must be clearly stated.

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1. Suppose K/k is a finite normal extension and let G be the group of automorphisms of K fixing every element of k. Denote by K^G the set of elements of K fixed by every element of G. Show that K^G/k is a purely inseparable extension.

2. Let k be a field and A, B finitely generated k-algebras. Let $f : A \to B$ be a k-algebra homomorphism. Show that if $\mathfrak{m} \subset B$ is a maximal ideal, $f^{-1}(\mathfrak{m}) \subset A$ is also a maximal ideal (Hint: Nullstellensatz). Recall that a k-algebra A is a ring homomorphism $k \to A$ sending the identity to the identity, and if A and B are k-algebras, a k-algebra homomorphism $A \to B$ is a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & \nearrow & \\ k & & \end{array}$$

3. (i) Let $\mathcal{C}$ be a category. Define what it means for an object X of $\mathcal{C}$ to be a *product* of a set of objects $\{X_i\}_{i\in I}$. (ii) Let $\mathcal{T}$ be the category of *torsion* abelian groups. Show that if $\{X_i\}_{i\in I}$ is any set of objects of $\mathcal{C}$, the product as defined in (i) exists. (iii) Give an example of a set $\{X_i\}_{i\in I}$ of objects of $\mathcal{T}$ whose product is not isomorphic to the usual product (i.e. in the category of abelian groups).

4. (i) Define what it means for a finite group G to be *nilpotent*. (ii) Show that if p is a prime and G is a p-group then G is nilpotent. (iii) Show that A_4 is solvable but not nilpotent.

5. Suppose R is an integral domain with fraction field K and I, J are invertible fractional ideals of R. Recall that the product IJ is the R-submodule of K generated by the products ab with $a \in I$ and $b \in J$. Show that $I \otimes_R J \simeq IJ$ as R-modules.

6. Suppose R is a commutative ring, $S \subseteq R$ is a multiplicative system (i.e. a nonempty subset closed under multiplication) and

$$0 \to L \to M \to N \to 0$$

is an exact sequence of R-modules. Show that the sequence

$$0 \to S^{-1}L \to S^{-1}M \to S^{-1}N \to 0$$

of R-modules is exact.

7. For any set S let $F(S)$ denote the free group on S. Which sets S have the property that $F(S)$ is an abelian group? Give a proof of your answer from first principles.

8. Suppose R is an integral domain. Show that the polynomial ring $R[X]$ is a Dedekind domain if and only if R is a field.

9. Suppose R is a noetherian ring and M is a finitely generated R-module. Show that M is noetherian.

10. Let R be a ring with identity. Show that any projective R-module is flat.

11. Let p be a prime. Classify up to isomorphism all semisimple rings of cardinality p^6.