PHD ALGEBRA EXAM SPRING 2023

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

(1) Briefly explain your answers to the following, freely using standard results.
  (a) (3 points) Find a polynomial $f(x) \in \mathbf{F}_3[x]$ such that $K = \mathbf{F}_3[x]/(f(x))$ is a field with 27 elements.
  (b) (4 points) Show that K is Galois over $\mathbf{F}_3$, and write down the automorphisms of K fixing $\mathbf{F}_3$.
  (c) (3 points) For which n is there an extension L/K of fields such that L has 3^n elements?

(2) Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree 5 with exactly three real roots. State and prove a result about the Galois group of the splitting field of $f(x)$ over $\mathbf{Q}$.

(3) Let $K = \mathbf{Q}(\zeta_{101})$, where ζ_{101} is a primitive 101th root of unity.
  (a) (4 points) Show that there is a unique subfield F of K such that $[F : \mathbf{Q}] = 2$.
  (b) (6 points) How many subfields of K contain F? Of these, which are Galois extensions of $\mathbf{Q}$?

(4) Recall that the Abelianization of a group G is $G^{\mathrm{ab}} := G/[G, G]$.
  (a) (5 points) Formulate and prove a universal property for abelianization.
  (b) (5 points) Show that abelianization is left adjoint to the forgetful functor from the category of abelian groups to the category of groups.

(5) Let K be a field and V a K-vector space. Recall that an alternating bilinear map $f : V \times V \to W$ is a function which is K-linear in each coordinate (i.e. K-multilinear) and such that $f(v, v) = 0$ for all $v \in V$.
  Show that the functor $\mathrm{Alt}^2_V : \mathrm{Vec}_K \to \mathrm{Sets}$ defined by

$$\mathrm{Alt}^2_V(W) = \{f : V \times V \to W : f \text{ is alternating bilinear}\}$$

is a representable functor. (The representing object, usually denoted $\Lambda^2(V)$, is an exterior power of V.) Suggestion: A quotient of $V \otimes_K V$

(6) Give examples of the following, and briefly explain your answers.
  (a) (4 points) a ring R and a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

  that is not split.
  (b) (3 points) a flat $\mathbf{Z}$-module which is not free.
  (c) (3 points) a non-zero injective $\mathbf{Z}$-module.

(7) Let R be a ring and M and N be R-modules. Let $P_\bullet$ and $P'_\bullet$ be projective resolutions of M and N. Given a morphsim of R-modules $f : M \to N$, show that there exists a morphism of resolutions $f : P_\bullet \to P'_\bullet$ extending f. In other words, show there exists $f_0, f_1, f_2, \ldots$ making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "f_0"] & M \arrow[d, "f"] \\
\cdots \arrow[r] & P'_2 \arrow[r, "d'_2"] & P'_1 \arrow[r, "d'_1"] & P'_0 \arrow[r, "\epsilon'"] & N
\end{tikzcd}$$

(8) What are the dimensions of $\mathbf{Q}[x]/(x^5) \otimes_{\mathbf{Q}} \mathbf{Q}[x]/(x^6)$ and of $\mathbf{Q}[x]/(x^5) \otimes_{\mathbf{Q}[x]} \mathbf{Q}[x]/(x^6)$ as $\mathbf{Q}$-vector spaces?

(9) Let F be a field. Prove that the power series ring

$$F[\![x]\!] = \left\{ \sum_{n=0}^{\infty} a_n x^n : a_n \in F \right\}$$

is a Noetherian ring.

(10) Let $B = \mathbf{Q}[x, y]/(xy)$ and $A = \mathbf{Q}[t]$. Note that both have Krull dimension 1. A homomorphism $f : A \to B$ of rings turns B into an A-module: for $a \in A$ and $b \in B$, $a.b = f(a)b$.

(a) (3 points) Explain why the Noether normalization lemma implies that there is a map $A \to B$ making B into a finitely generated A-module.

(b) (3 points) Show that the map $A \to B$ sending t to x does not make B into a finitely generated A-module.

(c) (4 points) Find an explicit map $A \to B$ that makes B into a finitely generated A-module and explain your answer.

(11) Let K be a field and consider the function $v : K(x) \to \mathbf{Z} \cup \{\infty\}$ given by

$$v(f/g) = \deg(g) - \deg(f), \quad v(0) = \infty.$$

(a) (5 points) Show that v is a discrete valuation on $K(x)$.

(b) (5 points) Show that the valuation ring $\mathcal{O}_v = \{R \in K(x) : v(R) \geq 0\}$ is a local ring. (Do not appeal to general facts about DVR's.)