# PHD ALGEBRA EXAM SPRING 2023

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

(1) Let $K$ be the splitting field of $x^7 - 2$ over $\mathbf{Q}$. What is $[K : \mathbf{Q}]$? Justify your answer.

(2) Give an example of a Galois extension of $\mathbf{Q}(\zeta_5)$ with degree 5 (where $\zeta_5$ is a primitive 5th root of unity). Give an example of a Galois extension of $\mathbf{Q}$ with degree 5. Justify your answers.

(3) Let $K$ be a field and $G$ a finite subgroup of $\mathrm{Aut}(K)$.
   (i) (2 points) What does the "theorem of the fixed field" say about $[K : K^G]$?
   (ii) (4 points) Use the theorem of the fixed field to show that if $K/F$ is a finite extension of fields then $\#\,\mathrm{Aut}(K/F) \le [K : F]$ with equality if and only if $F$ is the fixed field of $\mathrm{Aut}(K/F)$.
   (iii) (4 points) Use the theorem of the fixed field to show that if $F = K^G$ then $K/F$ is a Galois extension with Galois group $G$.

(4) Let Groups denotes the category of groups and Sets denotes the category of sets.
   (i) (5 points) Use free groups to construct a left adjoint to the forgetful functor $U : \text{Groups} \to \text{Sets}$. (You can skip checking "naturality" i.e. compatibility with morphisms.)
   (ii) (5 points) Show that the functor $F : \text{Groups} \to \text{Sets}$ defined by

$$F(G) = \{(x, y) \in G \times G : xyx^{-1}y^{-1} = 1\}$$

   is a representable functor. (Again, you can skip checking "naturality".)

(5) Let $k$ be a field. For a $k$-vector space $V$, recall that the dual vector space $V^*$ is defined to be the $k$-vector space of linear functionals on $V$, i.e. linear transformations from $V$ to $k$.
   (i) (5 points) Construct a natural linear transformation $\psi : V^* \otimes_k W^* \to (V \otimes_k W)^*$.
   (ii) (5 points) Show that $\psi$ is an isomorphism if $V$ and $W$ are finite dimensional.

(6) Let $R$ be a unital ring.
   (i) (4 points) State at least two equivalent definitions of a projective $R$-module. (You don't need to show they are equivalent, but you should include any definition you want to use in the next part.)
   (ii) (6 points) Let $M$ and $N$ be $R$-modules with $M \subset N$. If $M$ is projective and $N/M$ is projective, prove that $N$ is projective.

(7) Are the functors $F(A) = \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Q}/\mathbf{Z})$ and $G(A) = \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})$ from the category of $\mathbf{Z}$-modules to the category of $\mathbf{Z}$-modules exact? Justify your answer.

(8) Prove that the intersection of all prime ideals in a commutative ring is the nilradical. (Suggestion: for one direction, given a non-nilpotent element $a$ consider the set of all ideals not containing any power of $a$.)

(9) Let $\mathbf{R}$ and $\mathbf{C}$ denote the real and complex numbers as usual.
   (i) (3 points) Give an example of two different maximal ideals of $\mathbf{R}[x]$ which have the same zero set in the affine line $\mathbf{A}^1_{\mathbf{R}}$.
   (ii) (3 points) Give an example of two ideals of $\mathbf{C}[x]$ which have the same zero set in $\mathbf{A}^1_{\mathbf{C}}$.
   (iii) (4 points) State Hilbert's Nullstellensatz.

(10) Let $R$ be a commutative ring and $D$ a multiplicatively closed subset of $R$ containing 1. State the universal property for the localization $D^{-1}R$, show that any two rings satisfying the universal property are isomorphic, and construct a ring satisfying the universal property. (Checking the construction is well-defined will involve a variety of verifications which you can skip. But do check it satisfies the universal property.)

(11) Give an example of the following, and briefly justify your answers:
   (i) (5 points) A discrete valuation ring whose field of fractions is not of characteristic zero.
   (ii) (5 points) A Dedekind domain that is not a unique factorization domain.