Ph. D. Algebra Exam **August 23, 2022**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (a) (3 points) Give the definition of a free object in a concrete category $\mathcal{C}$.
   (b) (7 points) Let $\mathcal{G}$ be the category of finite groups. Prove that an object $H \in \mathcal{G}$ is free if and only if H is a trivial group.

2. (a) (2 points) Define what it means for a group G to be solvable.
   (b) (4 points) Let G be a solvable group and let N be a normal subgroup of G. Prove that G/N is solvable.
   (c) (4 points) Let F be a group which is free on $n > 1$ generators. Prove that F is not solvable.

3. Let m, n be positive integers. Find r such that there is an isomorphism of $\mathbb{Z}$-modules

$$(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}.$$

   Justify your answer.

4. Let R be a ring. Prove that the following are equivalent:
   (a) Every R-module is projective.
   (b) Every R-module is injective.
   (c) Every short exact sequence of R-modules splits.

5. Let G be a group, let F be a field, and let $\chi_1, \dots, \chi_n$ be distinct homomorphisms from G into $F^\times$. Prove that $\chi_1, \dots, \chi_n$ are linearly independent over F.

6. (a) (3 points) Let $n \geq 1$. Define the nth cyclotomic polynomial $\Phi_n(X)$.
   (b) (7 points) Prove that $\Phi_n(X)$ is irreducible over $\mathbb{Q}$. You may assume that $\Phi_n(X) \in \mathbb{Z}[X]$.

7. Let R be a commutative Noetherian ring with 1.

   (a) (2 points) Define what it means for a proper ideal $J \subset R$ to be irreducible.

   (b) (4 points) Prove that every irreducible proper ideal in R is primary.

   (c) (4 points) Let $I \subset R$ be an ideal. Prove that I has a primary decomposition.

8. Let S be a commutative ring with 1, let R be a subring of S which contains 1, and let $x \in S$. Suppose there is a subring $T \subset S$ such that $T \supset R[x]$ and T is finitely generated as an R-module. Prove that x is a root of some monic polynomial with coefficients in R.

9. Let R be a Noetherian integral domain with a unique nonzero prime ideal P. Assume that R is integrally closed in its fraction field F. Prove that P is a principal ideal.

10. Determine which of the following are Dedekind domains. Give a brief explanation in each case.

    (a) (2 points) $\mathbb{Z}[\sqrt{3}]$

    (b) (2 points) $\mathbb{Z}[\sqrt{5}]$

    (c) (3 points) $\mathbb{Q}[X]$

    (d) (3 points) $\mathbb{Q}[X, Y]$

11. Let D be a finite division ring. Prove that D is a field.