PhD Algebra Exam **January, 2022**

Answer seven problems, and on the list below circle the problems you wish to have graded. If more than seven problems are answered only the first seven will be graded. Write your answers clearly in complete English sentences.

Results from lectures or textbooks may be used without proof (within reason – don't state results equivalent to the problem), but must be clearly stated.

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1. Suppose K is a field of characteristic different from 2, $a \in \mathcal{K}$ is not a square and $b \in K$ is not a square in $K(\sqrt{a})$. Compute the degree of $L = K(\sqrt{a}, \sqrt{b})$ over K and determine all subfields of L containing K.

2. Suppose A is an integral domain with fraction field K, L/K is a field extension and $x \in L$. Show that x is integral over A if and only if there is a finitely generated A-module $M \subseteq L$ such that $xM \subseteq M$.

3. Suppose $\mathcal{C}$ is a category and X and Y are objects of $\mathcal{C}$ (i) (3 points) Define what it means for an object of $\mathcal{C}$ to be a *product* of X and Y. (ii) (7 points) Show that a product of X and Y is unique up to isomorphism.

4. Suppose G is a finite nilpotent group. Show that if H is a proper subgroup of G, it is a proper subgroup of its normalizer in G.

5. Let R be a integral domain. (i) (3 points) Define what it means for a fractional ideal of R to be *invertible*. (ii) (7 points) Show that an invertible ideal is a projective R-module.

6. Suppose R is a ring with identity (not necessarily commutative), M is a right R-module and
$$N' \to N \to N'' \to 0$$
is an exact sequence of left R-modules. Show that
$$M \otimes_R N' \to M \otimes_R N \to M \otimes_R N'' \to 0$$
is an exact sequence of abelian groups.

7. Suppose F is a free group on a set S, T is a subset of S and $H \subseteq F$ is the smallest normal subgroup of F containing T. Show that F/H is a free group on the set $S \setminus T$

8. Suppose R is a commutative ring with identity and I is an ideal contained in the Jacobson radical of R. Prove Nakayama's lemma: if M is a finitely generated R-module such that $IM = M$ then $M = 0$.

9. Suppose A is a noetherian commutative ring with identity. (i) Show that if $f : A \to B$ is a surjective homomorphism then B is noetherian. (ii) Recall that a *multiplicative system* in A is a nonempty subset closed under multiplication. Show that if $S \subset A$ is a multiplicative system then $S^{-1}A$ is noetherian.

10. Suppose R is a nontrivial commutative ring with identity and $S \subset R$ is a multiplicative system not containing 0 (see the previous problem for the definition). (i) (7 points) Show that there is a prime ideal $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$. (ii) (3 points) Use this to show that the intersection of all prime ideals of R is the set of nilpotent elements of R. Hint for (i): Apply Zorn's lemma on the ideals $I \subset R$ such that $R \cap S = \emptyset$.

11. (i) State the Jacobson density theorem for semisimple rings. (ii) Use it to show that if K is a field and A is a simple K-algebra of finite dimension then there is a division K-algebra D such that $A \simeq M_n(D)$ for some n.