PhD Algebra Exam **August, 2021**

Answer seven problems, and on the list below circle the problems you wish to have graded. If more than seven problems are answered only the first seven will be graded. Write your answers clearly in complete English sentences.

Results from lectures or textbooks may be used without proof (within reason – don't state results equivalent to the problem), but must be clearly stated.

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1. Suppose F is a finite field and K/F is a finite extension. Show that K/F is Galois and its Galois group is cyclic.

2. Let R be a ring with identity. (i) Define what it means for a left R-module to be *projective.* (ii) Show that a free R-module is projective. (iii) Give an example of a projective R-module that is not free.

3. (i) Let $\mathcal{C}$ be a category. Define what it means for an object of $\mathcal{C}$ to be the *product* of a set of objects $\{X_\alpha\}_{\alpha \in I}$. (ii) Let $\mathcal{C}$ be the category of torsion abelian groups (the full subcategory of the category of abelian groups whose objects are torsion abelian groups). Show that any set $\{X_\alpha\}_{\alpha \in I}$ has a product. (Hint: if I is infinite, it's *not* the usual product).

4. (i) Show that a discrete valuation ring is a Dedekind domain. (ii) Show that if F is a field, $F[X, Y]$ is not a Dedekind domain.

5. Suppose A is a domain with fraction field K, and assume A is integrally closed in K. Suppose $S \subseteq A$ is a multiplicative set. Show that $S^{-1}A \subseteq K$ is integrally closed in K.

6. Suppose A is a commutative ring with identity and $S \subseteq A$ is a multiplicative set. Let P be the set of prime ideals of A, P' the set of prime ideals $S^{-1}A$, and let $i : A \to S^{-1}A$ be the natural homomorphism. Show that $\mathfrak{p} \mapsto i^{-1}(\mathfrak{p})$ induces an injective map from P' to the subset of P consisting of $\mathfrak{q} \in P$ such that $\mathfrak{p} \cap S = \emptyset$.

7. Show that if X is a set with more than one element, the free group $F(X)$ on X is not abelian.

8. Let K be a field. Show that any vector space over K has a basis (do not assume the vector space has finite dimension).

9. Suppose R is a ring with identity, M is a left R-module and $N \subseteq M$ is a submodule. Show that if N and M/N are Noetherian then so is M.

10. Prove the Hilbert basis theorem: if A is a commutative Noetherian ring with identity then so is $A[X]$.

11. If p is a prime, describe all isomorphism classes of semisimple rings of order p^6.