PhD Algebra Exam **May, 2021**

Answer 7 questions from among the ones below. If you answer more than that, only the first seven will be graded.

Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

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1. Suppose that p is an odd prime and $K/\mathbb{Q}$ is the extension of $\mathbb{Q}$ obtained by adjoining a primitive pth root of 1 in $\mathbb{C}$. (i) (5 points) Show that K contains a unique quadratic extension F of $\mathbb{Q}$. (ii) (5 points) What is $\mathrm{Gal}(K/F)$?

2. Let G be a group, and recall that a *G-set* is a set S with a left G-action $G \times S \to S$. Write the action as $(g, x) \mapsto g.x$. A *morphism* of G-sets $f : S \to T$ is a map $f : S \to T$ such that $f(g.x) = g.f(x)$ for all $g \in G$ and $x \in S$. (i) (3 points) Show that G-sets and morphisms of G-sets form a category G-**Set**. (ii) (2 points) Show that there is a functor $H : G\text{-}\mathbf{Set} \to \mathbf{Set}$ which assigns to any G-set the underlying set. (iii) (5 points) Show that there is a functor $F : \mathbf{Set} \to G{-}\mathbf{Set}$ such that $F(S)$ is a free G-set on S (endow the set $G \times S$ with an appropriate G-action).

3. Suppose A is a ring with identity and I is a set. (i) (5 points) Show that if $\{P_\alpha\}_{\alpha\in I}$ is a set of left A-modules then $\bigoplus_\alpha P_\alpha$ is projective if and only if each P_α is projective. (ii) (5 points) Show that if $\{M_\alpha\}_{\alpha\in I}$ is a set of left A-modules then $\prod_{\alpha\in I} M_\alpha$ is injective if and only if M_α is injective.

4. Suppose $\mathcal{C}$ is a category and X and Y are objects of $\mathcal{C}$. (i) (3 points) Define what it means for an object Z of $\mathcal{C}$ to be a *coproduct* of X and Y. (ii) (7 points) Show that any two groups have a coproduct in the category of groups (if G and H are groups, construct the coproduct of G and H as a suitable quotient of the free group on the disjoint union $G \amalg H$).

5. Suppose K is a field, K^{alg} is an algebraic closure of K and L/K is a finite separable extension. Show that $L \otimes_K K^{alg}$ is a direct sum of fields. Hint: if $\alpha \in L$ has minimal polynomial $f \in K[X]$, observe that there is an exact sequence

$$0 \to K[X]f \to K[X] \to K(\alpha) \to 0.$$

6. Suppose A is a Dedekind ring and $S \subseteq A$ is a multiplicative system. Show that $S^{-1}A$ is a Dedekind ring.

7. Suppose A is a commutative ring with identity and $S \subseteq A$ is a multiplicative

system. (i) (6 points) Show that for any A-module M, $S^{-1}M \simeq S^{-1}A \otimes_A M$. (ii) (4 points) Show that $S^{-1}A$ is a flat A-module.

8. Prove the Hilbert basis theorem: if A is a commutative Noetherian ring then so is the polynomial ring $A[X]$.

9. Suppose A is an integral domain with fraction field K. (i) (3 points) Define what it means for an A-submodule $I \subseteq K$ to be an *invertible fractional ideal.* (ii) (7 points) Show that an invertible fractional ideal is a projective A-module.

10. Suppose A is a ring (not necessarily commutative, or with identity) and $J \subseteq A$ is its Jacobson radical (the intersection of all regular maximal left (or right) ideals). Show that if $J \neq A$ then A/J is semisimple.

11. Suppose K is a field and A is a *noncommutative* semisimple K-algebra of degree 16 (i.e. $\dim_K(A) = 16$). List representatives of every possible isomorphism class of A if (i) (3 points) $K = \mathbb{C}$, (ii) (4 points) $K = \mathbb{R}$, or (iii) (3 points) K is a finite field.