1st Semester Topology Exam

January, 2020

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Show that the set of functions $f : \mathbb{Z} \to \mathbb{Z}$ that are eventually 0 is countable.

2. Let $A \subset \mathbb{R}^2$ be a countable subset. Show that $\mathbb{R}^2 \setminus A$ is connected.

3. Show that every continuous map $f : [0, 1] \to [0, 1]$ has a fixed point. Is this true for continuous maps

(a) $f : [0, 1] \to (0, 1)$?

(b) $f : (0, 1) \to (0, 1)$?

4. Let X be connected, show that

(a) X^n is connected;

(b) X^ω is connected.

5. Let $f : X \to X$ be a map of a compact metric space to itself that satisfies the following condition: $d(f(x), f(y)) < d(x, y)$.

(a) Prove that f is continuous.

(b) Show that f has a fixed point and the fixed point is unique.

Answer the following with complete definitions or statements or short proofs.

6. State the Intermediate Value Theorem.

7. Does there exist a continuous surjective map from the 2-sphere S^2 to the interval $(0, 1)$?

8. Does there exist a continuous injective map from the 2-sphere S^2 to the interval $(0, 1)$?

9. State the Cantor-Schroeder-Bernstein Theorem.

10. Is every connected space path connected?

11. What is a basis of a topology? Does the set of all half-open intervals $\{(a, b], [c, d) \mid a < b,\ c < d\}$ form a basis of a topology on $\mathbb{R}$?

12. State the Extreme Value Theorem.

13. Is $\mathbb{R}^\omega$ connected in the uniform topology?

14. Is the set of integers $\mathbb{Z}$ is locally compact? Is the set of rational numbers $\mathbb{Q}$ locally compact?

15. Is $\mathbb{R}^2$ with the dictionary order topology metrizable?