Second Semester Algebra Exam **August 21, 2020**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6 7

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1. Suppose R is a commutative ring with identity and F is a free R-module of rank n. Show that for any R-module M the R-module $\operatorname{Hom}_R(F, M)$ is isomorphic to the direct sum of n copies of M.

2. Which of the following polynomials are irreducible? Explain, and factor the ones that are not irreducible.

   (a) (2 points) $X^3 + X + 1$ in $\mathbb{Q}[X]$;

   (b) (4 points) $X^4 + X^2 + 1$ in $\mathbb{Q}[X]$;

   (c) (4 points) $Y^2 - X^3 + X^2$ in $\mathbb{C}[X, Y]$.

3. Let R be an integral domain. (a) (2 points) Define what it means for an element of R to be *irreducible*. (b) (2 points) Define what it means for an element of R to be *prime*. (c) (6 points) Show that a prime element is irreducible.

4. Suppose F is a field, V is an F-vector space and $f : V \to V$ is a linear map. (a) (6 points) Show that if V has finite dimension then f is injective if and only if it is surjective. (b) (2 points) Give examples with V of infinite dimension, in which f is surjective but not injective (c) (2 points) Give an example in which f is injective but not surjective. Justify both examples; you may use any results on dimension if they are stated clearly.

5. A *minimal prime ideal* of a ring R is a prime ideal that does not properly contain any other prime ideal. If F is a field, find all minimal prime ideals in the ring
$$R = F[x_1, x_2, x_3, x_4, x_5, x_6]/(x_1x_2, x_3x_4, x_5x_6).$$

6. Suppose K is a field and $a \in K$ is such that $X^3 - a$ and $X^2 + X + 1$ are irreducible in $K[X]$. If $L = K[X]/(X^3 - a)$, show that L contains exactly one root of the polynomial $X^3 - a$

7. (a) (5 points) Find representatives in rational canonical form for all conjugacy classes in $GL_6(\mathbb{Q})$ with characteristic polynomial $(X^3 - 1)^2$. (b) For each representative, give the Jordan normal form in $GL_6(\mathbb{C})$.