First-year Analysis Examination
Part Two
May 2020

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $a < b$ and let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Prove that if $\int_a^b f = 0$ then for each positive integer n there exist $c < d$ in $[a, b]$ such that $\sup\{f(t) : c \leqslant t \leqslant d\} < 1/n$. Hence, or otherwise, prove that if $f > 0$ throughout $[a, b]$ then $\int_a^b f > 0$.

2. Let $(x_n : n \geqslant 0)$ converge in the metric space X and let $(f_n : n \geqslant 0)$ be a sequence of continuous real-valued functions on X. (a) Prove that if the sequence of functions is *uniformly* convergent, then $(f_n(x_n) : n \geqslant 0)$ is convergent. (b) Show by example that if *uniformly* is replaced by *pointwise* then $(f_n(x_n) : n \geqslant 0)$ may not converge.

3. Let $f : [0, 1] \to \mathbb{R}$ be continuous and assume that

$$\int_0^1 f(t)t^n \mathrm{d}t = 1/(n + 1)$$

for each integer $n > 1$. Deduce as much as is possible about f.

4. Let $(f_n : n \geqslant 0)$ be a sequence of measurable real-valued functions on some measurable space. Prove that each of the following sets is measurable:

$$A = \{\omega : f_n(\omega) \in [0, 1] \text{ for finitely many } n\}$$
$$B = \{\omega : f_n(\omega) \in [0, 1] \text{ for infinitely many } n\}$$
$$C = \{\omega : \text{the series } \sum_{n=0}^{\infty} f_n(\omega) \text{ is absolutely convergent}\}$$

5. For each positive integer n, let $f_n : [0, 1] \to [0, 1]$ be continuous; assume that $f_n \to 0$ pointwise as $n \to \infty$. Does it follow that

$$\int_0^1 f_n(t)\mathrm{d}t \to 0 \text{ as } n \to \infty?$$

Does your answer change if *continuous* is replaced by *Riemann integrable*?
