Answer **four** problems. (If you turn in more, the first four will be graded.) Put your answers in numerical order and circle the numbers of the four problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ____________________

Problems to be graded: 1 2 3 4 5 6

1. (a) (3 points) Define what it means for R to be a *principal ideal* domain.
   (b) (2 points) Give an example of an integral domain that is **not** a principal ideal domain.
   (c) (5 points) Let R be a principal ideal domain. Suppose that, for $i \in \mathbf{Z}$ with $i \geq 1$, I_i is an ideal of R, and $I_i \subseteq I_{i+1}$. Prove, from your definition of principal ideal domain, that there exists some $n_0 \in \mathbf{Z}$ with $n_0 \geq 1$ such that, for all $n \in \mathbf{Z}$, if $n \geq n_0$ then $I_n = I_{n_0}$.

2. (10 points) Let R be a non-zero commutative ring with identity. Prove that R has a maximal ideal.

3. Let R and S be integral domains and let $\phi\colon R \to S$ be a unital ring homomorphism. Let $\psi\colon R[x] \to S[x]$ be the corresponding ring homomorphism, i.e., ψ applied to a polynomial is the result of applying ϕ to each of its coefficients.
   (a) (7 points) Let $p(x) \in R[x]$ be non-constant and monic, and let $q(x) = \psi(p(x))$. Assume that $q(x) \in S[x]$ is irreducible. Prove that $p(x)$ is irreducible in $R[x]$.
   (b) (3 points) Prove that the converse need not be always true. More precisely, prove that if $p(x) \in R[x]$ is non-constant, monic and irreducible, and we set $q(x) = \psi(p(x))$, it need not be the case that $q(x) \in S[x]$ is irreducible.

4. (10 points) Let R be an integral domain and M a (unital) R-module. Suppose that M is finitely generated, and, for each $m \in M$, there exists some $r \in R$ such that $r \neq 0$ and $rm = 0$. Prove that, there exists some $a \in R$ such that $a \neq 0$ and, for all $m \in M$, $am = 0$.

5. (10 points) Let $R = \mathbf{Z}[\mathbf{i}]$ be the ring of Gaussian integers. Prove that R is a unique factorization domain. (You may assume standard results about different types of rings, but nothing, other than the definition and the fact that they form an integral domain, about Gaussian integers.)

6. (10 points) Let $n \geq 1$ and $F = \mathbf{Q}\left(\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n}\right)$. Prove that $2^{1/3} \notin F$.