First Semester Algebra Exam **August 21, 2020**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences.

Grade: 1 2 3 4 5 6 7

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1. Denote by Q_8 the quaternion group. Show that if $f : Q_8 \to S_n$ is an injective homomorphism then $n \geq 8$.

2. Let G be a group. (i) (5 points) Show that if G is abelian, the subset H of elements of finite order is a subgroup. (ii) (5 points) Give an example of a (nonabelian) group G and elements of finite order $x, y \in G$ such that xy has infinite order, and prove that this example is correct.

3. Prove the first assertion of Sylow's theorem: if G is a group of order n, p is prime and p^a is the largest power of p dividing n, then G has a subgroup of order p^a. State clearly any preliminary results you need.

4. (i) (3 points) Define what it means for a group G to be *nilpotent*. (ii) (4 points) Show that any nilpotent group is solvable. (iii) (3 points) Give an example of a finite solvable group that is not nilpotent, and prove that this example is correct.

5. Let G be a group acting on a set S, and $H \subseteq G$ a normal subgroup. Assume that for any $x_1, x_2 \in S$ there is a *unique* $h \in H$ such that $h(x_1) = x_2$. For $x \in S$ let G_x be the stabilizer of x. Show that for any $x \in S$, G is a semi-direct product of G_x and H. Do not assume that G is finite.

6. (1) (5 points) Find all conjugacy classes of elements of order 2 in S_6. (2) (5 points) Do the same for A_6. In both cases, justify your answer.

7. Show that the normalizer of a 5-Sylow subgroup of A_5 has order 10, and is a maximal subgroup of A_5.