First-year Analysis Examination
Part One
May 2020

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(a_n)_{n=0}^{\infty}$ be a bounded real sequence. Write L for the set of all real numbers ℓ such that $(\exists N)(\forall n > N)\, \ell < a_n$ and write U for the set of all real numbers u such that $(\exists N)(\forall n > N)\, a_n < u$. (i) Prove that $\sup L \leqslant \inf U$. (ii) Give an example of a *convergent* sequence for which the inequality in (i) is strict, or prove that no such sequence exists.

2. Let U be a subset of the metric space X. Does either of the following conditions imply the other? Prove or disprove, as appropriate. (i) U is open. (ii) For any $A \subseteq X$, if U meets the closure of A then U meets A itself.

3. Let X and Y be metric spaces. Decide whether the following statements are true or false. (i) If $(x_n)_{n=0}^{\infty}$ converges to x in X then $\{x\} \cup \{x_n : n \geqslant 0\}$ is compact. (ii) The function $f : X \to Y$ is continuous if its restriction $f|_K$ to each compact subset $K \subseteq X$ is continuous.

4. Let $f : X \to Y$ be a continuous bijection with inverse $g : Y \to X$ and assume that X is complete. (i) Show by example that if g is continuous then Y need not be complete. (ii) Prove that if g is *uniformly* continuous then Y must be complete.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and assume $\lim_{x\to\infty} f'(x) = L \in \mathbb{R}$. (i) When a is a positive real number, calculate

$$\lim_{x\to\infty} \frac{f(x+a) - f(x)}{a}.$$

(ii) Assume further that $f(x) \to A \in \mathbb{R}$ as $x \to \infty$ and calculate L.
