Department of Mathematics, University of Florida
First Semester Algebra Exam – January, 2019

Answer four problems. If you turn in more than four, only the first four will be graded. Unless otherwise indicated you may use theorems as long as you state them clearly. When you are done, put the answers in numerical order, put your name in the space below and circle the numbers of the problems you wish graded.

Name:

Problems: 1 2 3 4 5 6

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1. Suppose G is a p-group. (a) (5 points) Let S be a finite set on which G acts and denote by S^G the set of elements of S fixed by every element of G. Show that $|S^G| \equiv |S| \pmod{p}$. (b) (5 points) Use part (1) to show that if H is a nontrivial normal subgroup of G then $H \cap Z(G) \neq 1$ (consider the action of G on H by conjugation).

2. Suppose G is a group of order $231 = 3 \cdot 7 \cdot 11$. Show that G has a normal Sylow 7-subgroup and is isomorphic to the product of $\mathbb{Z}/11\mathbb{Z}$ and a solvable group of order 21.

3. Construct three nonisomorphic groups of order 56 whose Sylow 2-subgroup is isomorphic to D_8. (Use appropriate semidirect products, and observe that two semidirect products $K \rtimes_\phi H$, $K \rtimes_\psi H$ are nonisomorphic if $\mathrm{Ker}(\phi)$ and $\mathrm{Ker}(\psi)$ are not isomorphic).

4. Show that if G is a finite nilpotent group and H is a proper subgroup of G then H is a proper subgroup of its normalizer $N_G(H)$. (Argue by induction on $|G|$, and distinguish the cases $Z(G) \subseteq H$, $Z(G) \nsubseteq H$).

5. List up to isomorphism (a) the abelian groups of order p^4, (b) the number of elements of order p in each, and (c) the number of elements of order p^2 in each.

6. Suppose k and n are positive integers such that $k < n$. If $\sigma \in S_n$ is a k-cycle, show that $C_{S_n}(\sigma)$ is isomorphic to $(\mathbb{Z}/k\mathbb{Z}) \times S_{n-k}$.