1st Year Exam: Numerical Analysis, August, 2020.
Do **4** (four) problems.

1. Let $\{\phi_k\}_{k=0}^{n+1}$ be a set of orthogonal polynomials on $[a,b]$, with respect to the inner-product $(f,g) = \int_a^b f(x)g(x)w(x)\,\mathrm{d}\,x$, indexed so that ϕ_k is of degree k. Prove that ϕ_k has k distinct roots $\{x_j^{(k)}\}_{j=1}^k$, with $x_j^{(k)} \in [a,b]$, $j = 1,\dots,k$.

2. Consider the interval $[a,b]$ with the partition $a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b$. Suppose $s(x)$ is the natural cubic spline that interpolates the data $\{(x_i, y_i)\}_{i=1}^{n+1}$, and that $g \in C^2[a,b]$ interpolates the same data. Show that

$$\int_a^b (s''(x))^2 \,\mathrm{d}\,x \le \int_a^b (g''(x))^2 \,\mathrm{d}\,x.$$

3. **(a)** Consider the inner product on $C(0,2)$ given by $(f,g) = \int_0^2 f(t)g(t)\,\mathrm{d}\,t$. Find three orthonormal polynomials ϕ_0, ϕ_1, ϕ_2 on $(0,2)$ with respect to the given inner product such that the degree of ϕ_n is equal to n, $n = 0,1,2$.

(b) Find the nodes t_1 and t_2 and weights w_1 and w_2 which yield the weighted Gaussian Quadrature formula

$$\int_0^2 f(t)\,\mathrm{d}\,t \approx w_1 f(t_1) + w_2 f(t_2)$$

with degree of exactness $m = 3$. **You should find the nodes exactly, and may leave the weights w_1, w_2 in integral form**.

4. Prove for any $f \in C[a,b]$ and integer $n \ge 0$, that the best uniform approximation of f in P_n is unique. You may assume the existence of at least one best uniform approximation of f.

5. Suppose $f \in C^{n+1}[a,b]$, and let $p \in \mathcal{P}_n$ be a polynomial that interpolates the data $\{(x_i, f(x_i)\}_{i=0}^n$, where $x_0,\dots,x_n$, are distinct points in $[a,b]$. Consider an arbitrary fixed $x \in [a,b]$, and derive an exact expression for the error $f(x) - p(x)$.