1st Year Exam: Numerical Analysis, May, 2020.
Do **4** (four) problems.

1. Let $\alpha > 0$. For (i) $p = 2$; (ii) $p = \infty$, find the constant c_p that minimizes

$$E_p(c) = \|t^\alpha - c\|_p = \left(\int_0^1 |t^\alpha - c|^p \, \mathrm{d}\, t \right)^{1/p},$$

and find $E_p(c_p)$, for each of those values of p.

2. Let $\{\phi_k\}_{k=0}^{m+1}$ be the set of monic orthogonal polynomials with respect to inner product $(u, v) = \int_a^b u(x)v(x)w(x) \, \mathrm{d}\, x$. Let $\phi_{-1} = 0$ and $\phi_0 = 1$.

(a) Find expressions for constants α_k and β_k to determine a 3-term recurrence relation of the form

$$\phi_{k+1}(x) = (x - \alpha_k)\phi_k - \beta_k \phi_{k-1}(x), \; k = 0, \ldots, m.$$

(b) Use the above relation to determine ϕ_1, ϕ_2 and ϕ_3, using the inner-product $(u, v) = \int_0^1 u(x)v(x) \, \mathrm{d}\, x$.

3. Consider finding a zero of function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ that can be written as the sum of a linear and nonlinear part

$$f(x) = Bx + G(x),$$

where B is a nonsingular matrix and $G : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinear function. At a point x_k consider an affine model $M_k(x) = a_k + A_k(x - x_k)$, where the quantities $a_k \in \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n \times n}$ are to be determined.

(a) Determine a_k and A_k so that the following conditions hold:

$$M_k(x_k) = f(x_k) \quad \text{and} \quad M_k'(x_k) = B.$$

(b) Derive the iteration obtained by defining x_{k+1} as the zero of $M_k(x)$.

4. Let $f \in C^\infty(a, b)$. Let $x_0 < x_1 < x_2$ be three points in $[a, b]$ that are not necessarily equally spaced.

(a) Based the quadratic interpolant p_2 which satisfies $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$, find the centered finite difference approximations to $f'(x_1)$ and $f''(x_1)$ (you should explicitly show how the difference approximations are derived from the interpolant).

(b) Derive an expression for the error, $f'(x_1) - p_2'(x_1)$.

5. **(a)** For $f \in C^\infty(a, b)$, the composite trapezoidal quadrature rule with n subintervals with length $h = (b - a)/n$ satisfies

$$\left| \int_a^b f(x) \, \mathrm{d}\, x - I_{T,n} \right| = a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots,$$

where the coefficients $a_2, a_4, \ldots$, do not depend on n.

Find and inductively prove an expression for the kth Richardson extrapolant I_k of $I_0 := I_{T,n}$.

(b) Consider a quadrature formula of the type

$$\int_0^1 f(x)\,\mathrm{d}\,x \approx \alpha f(x_1) + \beta[f(1) - f(0)].$$

Determine α, β and x_1 such that the degree of exactness is as large as possible. What is the maximum degree of exactness?