1st Year Exam: Numerical Analysis, January, 2020.
Do **4** (four) problems.

1. **(a)** Starting with the first two Legendre polynomials over $[-1,1]$, given by $p_0(x) = 1$, and $p_1(x) = x$, find the next three (monic) Legendre polynomials, $p_2(x), p_3(x)$ and $p_4(x)$.

(b) Find the nodes t_0, t_1, t_2 and weights w_0, w_1, w_2 which define the Gaussian Quadrature formula

$$\int_{-1}^{1} f(t)\,\mathrm{d}\,t \approx w_0 f(t_0) + w_1 f(t_1) + w_2 f(t_2),$$

with degree of exactness $m = 5$. **You should find the nodes exactly, and may leave the weights w_0, w_1, w_2, in integral form**.

2. Consider the fixed point problem $x = f(x)$, where $f(x) = e^{-(3+x)}$.

(a) Assuming all computations are done in exact arithmetic, find the largest open interval in $\mathbb{R}$ where the fixed-point iteration $x_{k+1} = f(x_k)$ is ensured to converge.

(b) Write a Newton iteration for finding the fixed-point.

3. Let $\mathcal{P}_1$ be the space of polynomials of degree at most one. Using the norm $\|u\|_2 = \left(\int_a^b u^2\,\mathrm{d}\,x\right)^{1/2}$

(a) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a,b] = [-1,1]$.

(b) Find the least-squares approximation to $f(x) = x^4$ in $\mathcal{P}_1$ over $[a,b] = [0,1]$.

4. **(a)** Find a natural cubic spline B_0 on $-1 \le x \le 1$ that satisfies $v(-1) = 0$, $v(0) = 2$, and $v(1) = 0$.

(b) Find a natural cubic spline on $-1 \le x \le 2$ that satisfies $v(-1) = 0$, $v(0) = 2$, $v(1) = 1/2$ and $v(2) = 0$. You may write the solution as $B = B_0 + B_1$, where B_0 is (or is closely related to) the solution to **(a)**, if you like.

5. Let $f \in C^\infty(a-H, a+H)$, and let $h < H$. Let $x_0 = a-h$, $x_1 = a$ and $x_2 = a+h$.

(a) Find the finite difference approximation to $f''(a)$ based the quadratic interpolant p_2 which satisfies $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$ (you should explicitly show how the difference approximation is derived from the interpolant).

(b) Let $\psi_0(h) = \psi(h)$ be the difference approximation to $f''(a)$ found in part **(a)**. Assume (in exact arithmetic) $\psi(h) \to \psi(0) = f''(a)$ as $h \to 0$, and that $\psi(h)$ has the asympototic expansion

$$\psi(h) = \psi(0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots.$$

Find the general Richardson extrapolation formula for $\psi_k(h)$ based on $\psi_{k-1}(h)$ and $\psi_{k-1}(h/2)$.