1st Year Exam: Numerical Linear Algebra, January, 2020.
Do **4** (four) problems.

1. Let $A \in \mathbb{C}^{m\times n}$.

(a) Determine constants α and β such that the following inequality holds for the p and ∞ norms of matrix A, for integers $p \geq 1$. Justify your answer.

$$\alpha\|A\|_\infty \leq \|A\|_p \leq \beta\|A\|_\infty.$$

(b) Prove or give a counterexample: $\|A\|_2 \leq \|A\|_F$, where $\|A\|_F$ is the Frobenius norm of A. If you prove this, make sure to justify each nontrivial step.

2. Suppose A is Hermitian positive definite.

(a) Prove that each principal submatrix of A is Hermitian positive definite.

(b) Prove that an element of A with largest magnitude lies on the diagonal.

(c) Prove that A has a Cholesky decomposition.

3. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find both full and economy singular value decompositions of A.

(b) Find both full and economy QR decompositions of B.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

(a) If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

5. Suppose the linear equation $Ax = b$, with

$$A = \begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix}, \quad |\delta| < \epsilon_m/4,$$

is solved on a floating-point system with machine-epsilon ϵ_m, using an LU factorization (no pivoting!) followed by forward and back substitution. (You may assume the operations $(+, -, \div, \times)$, do not incur any additional errors).

(a) If $b = (1, 0)^T$, compute the backward error, $\|\hat{b} - b\|_2/\|b\|_2$, where $\hat{b}$ is the data that satisfies $A\hat{x} = \hat{b}$, and $\hat{x}$ is the computed solution.

(b) Is the result of **(a)** sufficient to draw any conclusions about the backward-stability of the algorithm used to compute $\hat{x}$? Explain.