First-year Analysis Examination
Part One
January 2020

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let f be a bounded real-valued function on the set $X \times Y$. Prove that

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leqslant \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Show by example that the inequality can be strict.

2. The *diameter* of the compact metric space X is defined by

$$\operatorname{diam} X = \sup\{d(p, q) : p, q \in X\}.$$

Prove that there exist $p_0, q_0 \in X$ such that $\operatorname{diam} X = d(p_0, q_o)$.

3. Let the function $f : X \to Y$ between metric spaces have closed graph $\{(x, y) : y = f(x)\} \subseteq X \times Y$. Prove the true and disprove the false:
(i) if X is compact then f is continuous;
(ii) if Y is compact then f is continuous.

4. Let $(s_n : n > 0)$ be a bounded real sequence and write $\sigma_n = (s_1 + \cdots + s_n)/n$ for each $n > 0$. Prove that

$$\limsup_n \sigma_n \leqslant \limsup_n s_n.$$

Hence, or otherwise, show that if $s_n \to s$ then $\sigma_n \to s$ also.

5. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $|f(x) - f(a)| \leqslant A|x - a|^{1+\alpha}$ for some constants $A, \alpha > 0$ and all $a, x \in \mathbb{R}$. Prove that f is constant.
