First-year Analysis Examination
Part Two
August 2020

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. The function $f : [0,1] \to \mathbb{R}$ is Riemann-integrable and vanishes at each point of a dense set. Does it follow that $\int_0^1 f = 0$? Proof or counterexample, as appropriate.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that there exists a sequence (p_n) of polynomials that converges to f both (i) pointwise on $\mathbb{R}$ and (ii) uniformly on each compact set in $\mathbb{R}$ (simultaneously).

3. Let $\mathcal{F} \subseteq C(X)$ be equicontinuous and let B be the set comprising all points of X at which $\mathcal{F}$ is bounded; that is, $a \in B$ exactly when there exists K such that $|f(a)| < K$ whenever $f \in \mathcal{F}$. Prove that $B \subseteq X$ is both closed and open.

4. Let (f_n) be a sequence of measurable real-valued functions on Ω. Prove that each of the following subsets of Ω is measurable:
(i) $P = \{\omega : (f_n(\omega))$ converges to an irrational number$\}$
(ii) $Q = \{\omega : (f_n(\omega))$ converges to a rational number$\}$
(iii) $R = \{\omega : (f_n(\omega))$ converges to a real number$\}$.

5. Let (f_n) be a sequence of bounded, measurable, real-valued functions on Ω and assume that $f_n \to f$ uniformly. Prove that if $\mu(\Omega) < \infty$ then

$$\int_\Omega f_n \mathrm{d}\mu \to \int_\Omega f \mathrm{d}\mu$$

and show by example that this can fail if $\mu(\Omega) = \infty$.
