MAA 5229 First-Year Exam, January 2020

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let f and $f_n\,(n > 0)$ be real-valued functions on $(0,1)$ with $f_n \to f$. Decide (with proof) the truth/falsity of each of the following statements when the convergence of f_n to f is (a) pointwise (b) uniform:

   (i) if each f_n is increasing on $(0,1)$ then so is f;

   (ii) if each f_n is bounded then so is f.

2. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function and let $\epsilon > 0$. Prove that there exist finitely many continuous functions $g_1, \ldots, g_n : [0,1] \to \mathbb{R}$ and $h_1, \ldots, h_n : [0,1] \to \mathbb{R}$ such that

$$\left| f(x,y) - \sum_{j=1}^{n} g_j(x)h_j(y) \right| < \epsilon \quad \text{for all } (x,y) \in [0,1] \times [0,1].$$

3. Let f_n be a sequence of nonnegative measurable functions and suppose there is an L^1 function g such that $f_n \leq g$ for all n. Prove that

$$\limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n.$$

   Give an example to show the conclusion can fail if the hypothesis $f_n \leq g$ is removed.

4. Let $(X, \mathscr{M})$ be a measurable space and $f_n : X \to \mathbb{R}$, $n = 1, 2, 3, \ldots$ a sequence of measurable functions. Prove that each of the following subsets of X is measurable:

   a) $\{x \mid f_n(x) \to +\infty\}$;

   b) $\{x \mid f_{n+1}(x) > f_n(x) \text{ for infinitely many } n\}$;

   c) $\{x \mid f_n(x) \text{ is rational for all } n\}$.

5. Let $E \subset [0,1]$ be a Lebesgue measurable set with $m(E) > 0$. Prove that there exists a point $0 < c < 1$ such that

$$m(E \cap [0,c]) = \frac{1}{2}m(E).$$