First-year Analysis Examination
Part One
August 2020

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let (a_n) and (b_n) be bounded real sequences. Define $\liminf a_n$ and $\limsup b_n$. Prove that

$$\liminf(a_n + b_n) \leq \liminf a_n + \limsup b_n \leq \limsup(a_n + b_n).$$

2. Let A be a subset of the metric space M. For each $t > 0$ let

$$(A)_t = \{x \in M : (\exists a \in A)\, d(x, a) < t\}$$

$$[A]_t = \{x \in M : (\exists a \in A)\, d(x, a) \leq t\}.$$

Is $\bigcap_{t>0}(A)_t$ equal to the closure of A? Is $\bigcap_{t>0}[A]_t$ equal to the closure of A? Proof or counterexample, as appropriate.

3. Let $f : [0, 1] \to [0, 1]$ be continuous and assume that for each positive integer n there exists $t \in [0, 1]$ such that $|f(t) - t| < 1/n$. Prove that there exists $a \in [0, 1]$ such that $f(a) = a$.

4. Let $f : X \to Y$ be a map between metric spaces, with graph

$$G = \{(x, y) : y = f(x)\} \subseteq X \times Y.$$

Prove that f is continuous if G is compact.

5. The function $f : (-1, 1) \to \mathbb{R}$ is differentiable and its derivative is bounded. Prove that there is a continuous function $F : [-1, 1] \to \mathbb{R}$ with the property that $f = F|_{(-1,1)}$.
