Numerical Analysis Exam-May 2024
Do **4** (four) problems

1. Consider an approximate integration scheme of the following form, for $0 < \alpha < 1$,

$$\int_0^1 x^\alpha f(x)dx \approx A \int_0^1 f(x)dx + B \int_0^1 xf(x)dx.$$

(a) Determine the constants A, B so that the formula has the maximum degree of exactness.

(b) What is the degree of exactness of the formula in (a)?

2. Let f be an arbitrary (continuous) function on $[0, 1]$ satisfying $f(x) + f(1 - x) \equiv 1$ for $0 \leq x \leq 1$.

(a) Show that $\int_0^1 f(x)dx = \frac{1}{2}$.

(b) Show that the composite trapezoidal rule for computing $\int_0^1 f(x)dx$ is exact.

3. Discuss uniqueness and non-uniqueness of the least squares approximation to a function f in the case of a discrete set $T = \{t_1, t_2\}(i.e., N = 2)$ where $t_1 \neq t_2$ and $\Phi_n = \mathcal{P}_{n-1}$ (polynomial of degree $\leq n - 1$). In case of non-uniqueness, determine all solutions.

4. Derive the three-point formula for the second derivative

$$f''(x) = \frac{1}{h^2}(f(x_0 - h) - 2f(x_0) + f(x_0 + h)) - \frac{h^2}{12} f^{(4)}(\eta),$$

where η is between $x_0 - h$ and $x_0 + h$.

5. Let $x_0, x_1, \cdots, x_n$ be pairwise distinct points in $[a, b]$, $-\infty < a < b < \infty$, and $f \in C^1[a, b]$. Show that given any $\epsilon > 0$, there exists a polynomial p such that $\|f - p\|_\infty < \epsilon$ and at the same time $p(x_i) = f(x_i)$, $i = 0, 1, \cdots, n$. Here $\|u\|_\infty = \max_{a \leq x \leq b} |u(x)|$.