First-Year Analysis Examination
August 2016 Part Two

Answer exactly FOUR questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

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1. Let the function $f : \mathbb{R} \to \mathbb{R}$ be *uniformly* continuous. For each positive integer n let $f_n(t) = f(t + \frac{1}{n})$ whenever t is a real number. Prove that the sequence $(f_n)_{n=1}^{\infty}$ is uniformly convergent. Give an example (with justification) to show that the conclusion can fail if the hypothesis *uniformly* is dropped.

2. Let $A = \{a_n : n \geqslant 1\}$ be a countably infinite subset of $[0, 1]$ and let $1_A : [0, 1] \to \mathbb{R}$ be its indicator (or characteristic) function. Exhibit such a set A for which 1_A *is not* Riemann integrable and exhibit such a set A for which 1_A *is* Riemann integrable, providing justification in each case.

3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from $[0, 1]$ to $[0, 1]$. Prove that if $f_n(t)$ *decreases* to 0 whenever $0 \leqslant t \leqslant 1$ then $\int_0^1 f_n(t)\mathrm{d}t \to 0$. Does the same conclusion follow when *decreases* is replaced by *converges*? Justify.

4. Prove that if the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable then its derivative is measurable.

5. Let f be a Lebesgue integrable function on $\mathbb{R}$ and let g be defined on $\mathbb{R}$ by $g(y) = \int_{-\infty}^{\infty} \cos(xy)f(x)dx$. Prove that $\lim_{k\to\infty} g(k) = 0$.