First Year Exam (Algebra, Semester 2), May 2022

Answer 4 questions. You should indicate which 4 problems you wish to be graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Prove that the following polynomials are irreducible in $\mathbb{Q}[x]$, stating clearly any theorems you wish to apply.
   (a) (3 points) $x^4 - 75$.
   (b) (3 points) $x^3 - 5x^2 + x + 5$.
   (c) (4 points) $x^4 - 14x^3 + 2x^2 + 21x - 7$.

2. (10 points) Let R be the quotient of the ring $\mathbb{Z}[i]$ of Gaussian integers by the principal ideal $(3 + 3i)$. Show that R is finite and determine the number of its elements.

3. Let R be a commutative ring with 1 and let I, J and P be ideals in R such that $I \cap J \subseteq P$.
   (a) (5 points) Prove that if P is a prime ideal then either $I \subseteq P$ or $J \subseteq P$.
   (b) (5 points) Give an example which shows that if P is not prime then the conclusion above may not hold.

4. Let R be a ring with 1 and M a (unital) left R-module.
   (a) (2 points) State what it means for M to satisfy the Ascending Chain Condition (i.e. for M to be Noetherian).
   (b) (4 points) Prove that if M satisfies the ACC then every submodule of M is finitely generated.
   (c) (4 points) Give an example of a finitely generated module that does not satisfy the ACC.

5. (10 points) Determine the conjugacy classes of elements of order 8 in $\mathrm{GL}(4, \mathbb{Q}(\sqrt{2}))$. (Hint: Factorize $x^4 + 1$ in $\mathbb{Q}(\sqrt{2})[x]$.)

6. Let E be an extension field of the field F.
   (a) (2 points) What does it mean for E to be an algebraic extension of F?
   (b) (4 points) Show that if α and β are elements of E that are both roots of the same irreducible polynomial in $F[x]$, then the subfields $F(\alpha)$ and $F(\beta)$ of E are isomorphic.
   (c) (4 points) Give an example where the subfields in (b) are not equal.