First-Year Second-Semester Exam
May 2025

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Let $p\colon E \to B$ be a covering map, and let $p(e_0) = b_0$. Define the map $\Phi\colon \pi_1(B, b_0) \to p^{-1}(b_0)$ which gives the the lifting correspondence. Prove that Φ is surjective if E is path connected, and bijective if E is simply connected.
2. Prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$.
3. (a) Define the Hawaiian earring space, H.
   (b) Show the images of the generators of the fundamental groups of the circles do not generate the fundamental group of H.
4. (a) Define surface.
   (b) Compute the fundamental group of the connected sum of a torus and second torus $T^2 \# T^2$.
5. Let X be the quotient space obtained from an 8-sided polygonal region P by pasting its edges together according to the labeling scheme $abcdc^{-1}a^{-1}db$. It turns out that all vertices of P are mapped to the same point of the quotient space X by the pasting map. Calculate $H_1(X)$, and using this, determine which compact surface X is homeomorphic to.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. State the Tietze Extension Theorem.
7. Show that if $h, h'\colon X \to Y$ are homotopic and $k, k'\colon Y \to Z$ are homotopic, then $k \circ h$ and $k' \circ h'$ are homotopic.
8. Let E and B be topological spaces. Define what it means for a function $p\colon E \to B$ to be a covering map.
9. If $j : A \hookrightarrow X$ is the inclusion of a retract, then show that the induced map on fundamental groups is injective.
10. Let α be a path in X from x_0 to x_1. Define the map $\hat{\alpha}\colon \pi_1(X, x_0) \to \pi_1(X, x_1)$ and show that $\hat{\alpha}$ is a group isomorphism.
11. State the Seifert-van Kampen Theorem.
12. Let S^n be the n-sphere. Prove that $\pi_1(S^n)$ is trivial for $n \geq 2$.
13. State the Borsuk-Ulam Theorem for S^2.
14. State the Brouwer Fixed Point Theorem for the disk.
15. Give an example of a topological space whose fundamental group is $\mathbb{Z}/3\mathbb{Z}$. Give an example of a topological space whose fundamental group is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.