First-Year Second-Semester Exam
January 2025

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Suppose that the continuous maps $f, g : X \to Y$ are homotopic and that the continuous maps $h, k : Y \to Z$ are homotopic. Show that $h \circ f$ is homotopic to $k \circ g$.
2. Prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$.
3. State and prove the Brouwer fixed point theorem for the unit disk, D^2.
4. Prove that for $n \geq 2$, the fundamental group of the n-sphere, S^n, is 0.
5. Show that the torus $T = S^1 \times S^1$ is not homeomorphic to the surface of genus two, $M_2 = T\#T$.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. Suppose that $f, g : X \to \mathbb{R}^n$ are continuous. Show that f and g are homotopic.
7. State the Borsuk-Ulam Theorem for S^2.
8. State the Seifert-van Kampen Theorem.
9. State the Tietze Extension Theorem.
10. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.
11. State the Tychonoff theorem.
12. Describe a space X with $\pi(X) \cong \mathbb{Z}/3\mathbb{Z}$.
13. If $j : A \hookrightarrow X$ is the inclusion of a retract, then show that the induced map on fundamental groups $j_\#$ is injective.
14. Give the definition of a deformation retraction.
15. Define Stone-Čech compactification.