First Year Topology Exam, Second Semester
May 2024

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Let $p\colon E \to B$ be a covering map, and let $p(e_0) = b_0$. Prove that the lifting correspondence $\phi\colon \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective if E is path connected, and bijective if E is simply connected.

2. Show that a continuous map $f\colon S^1 \to S^1$ is homotopic to $z^n\colon S^1 \to S^1$ (given by $e^{i\theta} \mapsto e^{in\theta}$) for some integer $n \in \mathbb{Z}$.

3. State and prove the Brouwer fixed point theorem for the unit disk, D^2.

4. The connected sum of two surfaces S and S' is obtained by cutting out a disk from each surface and gluing the resulting surfaces along their circle boundaries: $S\#S' = (S \setminus D^2) \cup_{S^1} (S' \setminus D^2)$. Let T be the torus. Using the Seifert-van Kampen theorem on the natural decomposition into two pieces given by the connected sum, calculate $\pi_1(T\#T)$.

5. Prove that if X is path connected and x_0 and x_1 are two points of X, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Answer the following problems with complete definitions, complete statements, an example, or a short proof. (Each problem 5 points.)

6. State the Seifert-van Kampen theorem.

7. Show that the Tietze Extension Theorem implies Urysohn's Lemma.

8. Let X be a topological space. Define what it means for X to be *compact*.

9. Let X be a completely regular space. Define the Stone-Čech compactification of X.

10. State the Tychonoff Theorem.

11. Let $[0,1]^J$ be the set of all functions from a set J to $[0,1]$. Prove that if $\mathcal{F}$ is a closed subset of $[0,1]^J$, then $\mathcal{F}$ is compact.

12. Let $A \subset X$. Let $r\colon X \to A$ be a *retraction*, meaning r is continuous and $r(a) = a$ for all $a \in A$. Show that for any $a_0 \in A$, the induced map $r_*\colon \pi_1(X, a_0) \to \pi_1(A, a_0)$ on the fundamental groups is surjective.

13. Let $\gamma\colon [0,1] \to X$ be a loop with basepoint x_0. Let γ^{-1} be defined by $\gamma^{-1}(t) = \gamma(1-t)$. Show that $\gamma * \gamma^{-1}$ is loop homotopic to the constant loop.

14. Describe a surface whose fundamental group is not abelian.

15. Let X and Y be topological spaces and let $f\colon X \to Y$ be continuous and surjective. Show that if X is connected, then so is Y.