1st Year 2nd Semester Topology Exam
August, 2023

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate page.

1. Show that B^2 is not homeomorphic to B^3 where B^n denotes the closed unit ball in $\mathbb{R}^n$.
2. Show that every continuous map $f : S^2 \to S^1$ of the 2-sphere to the circle S^1 is null-homotopic.
3. Let

$$X = ([-1, 1]\times\{0\}\times\{0\})\cup(\{0\}\times[-1, 1]\times\{0\})\cup(\{0\times\{0\}\times[-1, 1]) \subset \mathbb{R}^3.$$

Show that every continuous map $f : X \to X$ has a fixed point.
4. Derive the Brouwer Fixed Point Theorem for B^2 from the Borsuk-Ulam Theorem for S^2.
5. Let X, Y, and Z denote the following open rays in $\mathbb{R}^3$:

$$X = (0, \infty)\times\{0\}\times\{0\},\ Y = \{0\}\times(0, \infty)\times\{0\},\ Z = \{0\}\times\{0\}\times(0, \infty)$$

Compute $\pi_1(A)$ where
(a) $A = \mathbb{R}^3 \setminus X$?
(b) $A = \mathbb{R}^3 \setminus \bar{X}$? Here $\bar{X}$ denotes the closure of X in $\mathbb{R}^3$.
(c) $A = \mathbb{R}^3 \setminus (X \cup Y)$?
(d) $A = \mathbb{R}^3 \setminus (\bar{X} \cup Y)$?
(e) $A = \mathbb{R}^3 \setminus (X \cup Y \cup Z)$?
(f) $A = \mathbb{R}^3 \setminus (\bar{X} \cup Y \cup Z)$?

Answer the following with complete definitions or proofs.

6. Describe a space X with $\pi_1(X) \cong \mathbb{Z}_3$
7. Show that the projective plane $\mathbb{R}P^2$ is not homeomorphic to the surface of genus 2, $M_2 = T\#T$.
8. Let $X = [0, 1]^{(0,1)}$ be given the product topology. Is the space X
(a) compact?
(b) metrizable?
(c) separable?
9. State the Seifert-van Kampen Theorem. Can it be used to compute $\pi_1(S^n)$?
10. State the Tychonoff Theorem. Prove that every compact Hausdorff space X can be embedded into a Tychonoff cube $[0, 1]^A$.