2nd Semester Topology Exam

May 2023

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper (10 pts each problem).

1. Show that every continuous map $f : S^1 \to S^2$ of the circle to the 2-sphere is null-homotopic.

2. Show that the Stone-Čech compactification $\beta(\mathbb{N})$ of the naturals is uncountable.

3. Let $X_{\geq 0}$, $Y_{\geq 0}$, and $Z_{\geq 0}$ denote the nonnegative parts of the x-axis, the y-axis, and the z-axis in $\mathbb{R}^3$ respectively. Let $A = \mathbb{R}^3 \setminus (X_{\geq 0} \cup Y_{\geq 0} \cup Z_{\geq 0})$. Compute the fundamental group $\pi_1(A)$.

4. Let $f : S^1 \to T = S^1 \times S^1$ be a map to a torus defined by the formula $f(z) = (z, -z)$. Compute the fundamental group $\pi_1(X)$ where $X = T \cup_f B^2$ is obtained from T by attaching a 2-disk along f.

5. Show that the projective plane $\mathbb{R}P^2$ is not homeomorphic to the Klein bottle K. What about the connected sum $\mathbb{R}P^2 \# \mathbb{R}P^2$?

Answer the following with complete definitions or statements or short proofs (5 pts each problem).

6. Show that S^2 is not homeomorphic to S^3.

7. Let CX denote the cone over the n-point space $X = \{1, \dots, n\}$. Show that every continuous map $f : CX \to CX$ has a fixed point.

8. State the Borsuk-Ulam Theorem for S^2.

9. Show that if $g : S^2 \to S^2$ is continuous and $g(x) \neq g(-x)$ for all x, then g is surjective.

10. State the Tychonoff theorem.

11. State the Urysohn Lemma.

12. State the Seifert-van Kampen Theorem. Can it be used to compute the fundamental group of S^1?

13. Show that for a retraction $r : X \to A$ for any $a_0 \in A$ the induced map $r_* : \pi_1(X, a_0) \to \pi_1(A, a_0)$ on the fundamental groups is surjective.

14. Give definition of a deformation retraction. Let A be a deformation retract of X. Show that the inclusion map $A \to X$ is a homotopy equivalence.

15. Is I^I separable?