1st Year 2nd Semester Topology Exam

January, 2023

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Show that every map $f : S^2 \to T$ of the 2-sphere to the torus T is null-homotopic.
2. Show that the torus T is not homeomorphic to the surface of genus 2, $M_2 = T \# T$.
3. Show that for every continuous map $f : S^1 \to S^1$ there is $n \in \mathbb{Z}$ such that f is homotopic to $z^n : S^1 \to S^1$.
4. Let $X = [-1, 1] \times \{0\} \cup \{0\} \times [-1, 1] \subset \mathbb{R} \times \mathbb{R}$. Show that every continuous map $f : X \to X$ has a fixed point.
5. Let X, Y, and Z denote the x-axis, the y-axis, and the z-axis in $\mathbb{R}^3$. Is $\pi_1(A)$ abelian where
   (a) $A = \mathbb{R}^3 \setminus X$?
   (b) $A = \mathbb{R}^3 \setminus (X \cup Y)$?
   (c) $A = \mathbb{R}^3 \setminus (X \cup Y \cup Z)$?

Answer the following with complete definitions or statements or proofs.

6. Show that S^2 is not homeomorphic to S^3.
7. State the Borsuk-Ulam Theorem for S^2.
8. Let $I = [0, 1]$ and let $X = I^I$ be given the product topology. Is the space X
   (a) compact?
   (b) metrizable?
   (c) separable?
9. State the Seifert-van Kampen Theorem. Can it be used to compute $\pi_1(S^1)$?
10. (a) What are completely regular spaces?
    (b) Define the Stone-Čech compactification βX of X.