TOPOLOGY FIRST YEAR EXAM PART 2 • SPRING 2022

Instructions: Do seven **7** and only seven of the following. In the following D^2 is the closed, two-dimensional disk, and S^1 is the unit circle, its boundary. The cyclic group of order n, is $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

1. Assume that X is completely regular. Show that X is connected if and only if βX is connected.

2. a. Define a retract and show there is no retract $D^2 \to S^1$

   b. Show that any continuous map $f : D^2 \to D^2$ has a fixed point.

   c. If A is retract of D^2 prove that any continuous map $f : A \to A$ has a fixed point.

3. Let M and N be two connected, compact manifolds with points $x_0 \in M$ and $y_0 \in N$. Define an equivalence relation on their disjoint union, $M \sqcup N$, as $x_0 \sim y_0$ and all other points are just equivalent to themselves. Let $M \vee N = (M \sqcup N)/\sim$. Show that

$$\pi_1(M \vee N, [x_0]) = \pi_1(M, x_0) * \pi_1(N, y_0)$$

   where $[x_0]$ is the equivalence class of x_0. ,

4. Construct a space whose fundamental group is isomorphic to $(\mathbb{Z} \oplus \mathbb{Z}_3) * \mathbb{Z}_5$

5. Show that S^2 is not homeomorphic to S^n for any $n > 2$.

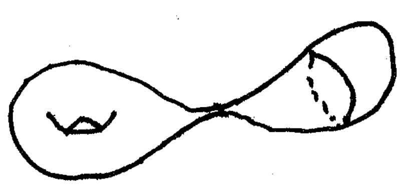
6. State the Seifert-van Kampen Theorem and use it to compute the fundamental group of the space pictured below.

7. The projective plane PP is constructed by identifying the edges of a disk as pictured below. Compute the fundamental group of PP as well as that of PP minus n points, $n > 0$.

8. (a) Define a deformation retract and show it induces an isomorphism on fundamental groups.

   (b) Prove or disprove: a retract induces an isomorphism on fundamental groups.

#6



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