**1st Year 2nd Semester Topology Exam**
January, 2022

**Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper. Each is worth 15 points.**

1. Show that the torus $T$ is not homeomorphic to the surface of genus 2, $M_2 = T\#T$.

2. Prove that the set of irrational numbers $\mathcal{I} \subset \mathbb{R}$ cannot be presented as a countable union of closed sets each of which has empty interior in $\mathcal{I}$.

3. Prove that $\pi_1(S^1) = \mathbb{Z}$.

4. Let $CX$ denote the cone over the $n$-point space $X = \{1, \dots, n\}$. Show that every continuous map $f : CX \to CX$ has a fixed point.

5. Show that the fundamental group of $M = K\#T$ is not abelian where $K$ is the Klein bottle.

**Answer the following with complete definitions or statements or proofs. Each is worth 5 points.**

6. Show that every map $f : S^2 \to \prod_{\alpha \in J} S^1$ of the 2-sphere to the product of circles is null-homotopic.

7. Show that for a retraction $r : X \to A$ the induced map $r_\#$ on fundamental groups is surjective.

8. Which of the following spaces taken with the product topology are locally compact? separable ?

(a) $\mathbb{N}^{\mathbb{N}}$;

(b) $I^{\mathbb{N}}$;

(c) $N^{\mathbb{I}}$;

(d) $I^I$.

Here $I = [0, 1]$ and $\mathbb{N}$ is the set of natural numbers.

9. State the Seifert-van Kampen Theorem. Can it be used to compute the fundamental group of $S^1$?

10. Show that $S^2$ is not homeomorphic to $S^4$