First-Year Second-Semester Exam
May 2021

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 15 points.

1. (a) State the Urysohn Lemma.
   (b) Let X be a connected normal space having two distinct points x and y. Show that X has uncountably many points.
2. (a) State the Brouwer fixed-point theorem for the disc B^2.
   (b) Let A be a 3 by 3 matrix of positive real numbers. Show that A has a positive real eigenvalue. Hint: Let B be the intersection of the unit sphere S^2 and the positive octant, $\{(x, y, z \in \mathbb{R}^3 \mid x, y, z \geq 0\}$.
3. Prove that $\pi_1(S^1, 1) \cong \mathbb{Z}$.
4. (a) Define the Hawaiian earring space, H.
   (b) Show the images of the generators of the fundamental groups of the circles do not generate the fundamental group of H.
5. (a) Define surface.
   (b) Compute the fundamental group of the connected sum of a torus and second torus $T^2 \# T^2$.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. Show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.
7. If $j : A \hookrightarrow X$ is the inclusion of a retract, then show that the induced map on fundamental groups $j_\#$ is injective.
8. State the *Seifert-van Kampen Theorem.*
9. What is the fundamental group of the torus $T^2 = S^1 \times S^1$?
10. Given continuous maps of topological spaces $f : A \to X$ and $g : A \to Y$ state the universal property satisfied by the pushout.