First-Year First-Semester Exam
January 2025

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 10 points.

1. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto. Note that 2^X is the power set on X and is also denoted by $\mathcal{P}(X)$.

2. (a) Define what it means for τ to be a topology on a set X.

   (b) If $\{\tau_\alpha\}_{\alpha \in J}$ is a family of topologies on X, show that $\bigcap_{j \in J} \tau_\alpha$ is a topology on X. (*Remark:* $\bigcap_{j \in J} \tau_\alpha = \{U \mid U \in \tau_\alpha \ \forall \alpha \in J\}$.)

3. Let $p : X \to Y$.

   (a) Define what it means for p to be a quotient map.

   (b) Assume that p is continuous. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

4. Let $f, g : X \to Y$ be continuous maps to a Hausdorff space. Prove that the set $\{x \mid f(x) = g(x)\}$ is a closed subset of X.

5. Prove that if X is a compact Hausdorff space, then X is a regular space.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. Prove that $\{(a, b] \mid a < b\}$ is a basis for a topology on $\mathbb{R}$.

7. Let (X, τ) be a topological space. Let $A \subset X$. Give the definition of the subspace topology on A.

8. Suppose that $f : X \to Y$ is continuous and that $A \subset X$ is connected. Show that $f(A) \subset Y$ is connected.

9. Does a connected space need to be path connected?

10. State the Intermediate Value Theorem.

11. Describe the one-point compactification of $(0, 1) \cup (2, 3)$.

12. State the Urysohn Lemma.

13. Show that a connected normal space X having more than one point is uncountable.

14. What does it mean for a topological space X to be a Baire space?

15. Show that if $A \subset \mathbb{R}$ is connected then A is an interval.