First-Year First-Semester Exam
January 2025

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 10 points.

1. Show that for every set X, there is no function $f : X \to 2^X$ such that f is onto. Note that 2^X is the power set on X and is also denoted by $\mathcal{P}(X)$.

2. Let $\mathbb{R}^\infty$ denote the subset of $\mathbb{R}^\omega$ consisting of all sequences that are "eventually zero," that is, all sequences $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbb{R}^\infty$ in the box and product topologies? Justify your answer.

3. Let $p : X \to Y$.
   (a) Define what it means for p to be a quotient map.
   (b) Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y, then p is a quotient map.

4. (a) Give the topology of the one-point compactification of a locally compact Hausdorff topological space X.
   (b) Show that the one-point compactification of $\mathbb{Z}_+$ is homeomorphic with the subspace $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ of $\mathbb{R}$.

5. Let (X, d) be a complete metric space. State and prove the contraction mapping theorem for X, which is also called the Banach fixed point theorem.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. Show that the axiom of choice is equivalent to the statement that for any indexed family $\{A_\alpha\}_{\alpha \in J}$ of nonempty sets with $J \neq \emptyset$, the cartesian product $\prod_{\alpha \in J} A_\alpha$ is nonempty.

7. Show that a connected normal space X having more than one point is uncountable.

8. Suppose that $f : X \to Y$ is continuous and that $A \subset X$ is connected. Show that $f(A) \subset Y$ is connected.

9. Suppose that $f : X \to Y$ is continuous and that $A \subset X$ is compact. Show that $f(A)$ is compact.

10. State the Extreme Value Theorem.

11. Define what it means for a topological space to be regular.

12. What does it mean for a topological space X to be a Baire space?

13. State the Cantor-Schröder-Bernstein theorem.

14. Is every path connected space locally path connected?

15. Define what it means for a topological space X to be separable. Let I denote the unit interval $[0, 1]$. Is $I^{\mathbb{Z}_+}$ separable? What about I^I?