First Year Topology Exam, First Semester
August 2024

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Let X be a set. Show there is no bijection between X and $\mathcal{P}(X)$, the power set of X.

2. Show that $\mathbb{R}^\omega$ with the box topology is not connected.

3. Show that if X is a compact Hausdorff space, then X is a regular space.

4. Let A be a partially ordered set. The maximum principle states that there is a maximal totally ordered subset B of A. Zorn's lemma states that if every totally ordered subset of A has an upper bound in A, then A has a maximal element. Prove that the maximum principle implies Zorn's lemma.

5. Let (X, d) be a metric space. Suppose there exists some $\varepsilon > 0$ such that for all $x \in X$, the closure $\overline{B_\varepsilon(x)}$ of the ε-ball about x is compact. Prove that X is complete.

Answer the following problems with complete definitions, complete statements, an example, or a short proof. (Each problem 5 points.)

6. Define what it means for a topological space X to be *normal*.

7. Let X be compact, and let $C_1, C_2, C_3, \ldots$ be a sequence of closed sets in X. If $\cap_{i=1}^{k} C_i$ is nonempty for all $k \geq 1$, then prove that $\cap_{i=1}^{\infty} C_i$ is nonempty.

8. Let $\{A_n\}_{n\in\mathbb{Z}_+}$ be a countable collection of countable sets. Prove that the union $\cup_{n\in\mathbb{Z}_+} A_n$ is countable.

9. Show that $[0, 1]$ and $[0, 1)$ are not homeomorphic.

10. Show that $f\colon \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(x) = (x, x, x, \ldots)$ is not continuous if $\mathbb{R}^\omega$ has the box topology.

11. Give an example of a topological space that is connected but not locally connected.

12. State the Baire Category Theorem.

13. Let (x_n) be a Cauchy sequence in the metric space X. Show that if (x_n) has a subsequence (x_{n_k}) that converges, then (x_n) also converges.

14. Give an example of a metric space (X, d) and a continuous function $f\colon X \to X$ satisfying (i) $d(f(x), f(y)) < d(x, y)$ for all $x, y \in X$ and (ii) $f(x) \neq x$ for all $x \in X$.

15. Let X be a topological space. Suppose there is a space Y such that (i) X is a subspace of Y, (ii) $Y \setminus X$ is a single point, and (iii) Y is compact Hausdorff. Prove that X is locally compact.