First Year Topology Exam, First Semester
May 2024

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Prove that $\{0,1\}^\omega$ is not countable.

2. Show that $\mathbb{R}^\omega$ with the box topology is not connected.

3. Assume that $f\colon X \to Y$ is continuous and surjective.

   (a) If X is Lindelöf (every open covering has a countable subcovering), show that Y is also.

   (b) If X is separable (has a countable dense subset), show that Y is also.

4. Let (X, d) be a compact metric space. Let $f\colon X \to X$ and suppose there is some $c < 1$ with $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Prove there exists a unique $x \in X$ with $f(x) = x$.

5. Let (Y, d) be a complete metric space. Let J be a set. Suppose $d(y, y') \leq 1$ for all $y, y' \in Y$, meaning a simpler definition for the uniform metric $\overline{\rho}$ on Y^J is $\overline{\rho}(f, g) = \sup_{\alpha \in J}\{d(f(\alpha), g(\alpha))\}$ for $f, g\colon J \to Y$. Prove that $(Y^J, \overline{\rho})$ is a complete metric space.

Answer the following problems with complete definitions, complete statements, an example, or a short proof. (Each problem 5 points.)

6. Define what it means for a topological space X to be *regular*.

7. Show that a closed subspace C of a compact space X is compact.

8. A function $f\colon \mathbb{Z}_+ \to \{0,1\}$ is *eventually zero* if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$. Prove that the set of all functions $f\colon \mathbb{Z}_+ \to \{0,1\}$ that are eventually zero is countable.

9. Show that $[0, \infty)$ and $\mathbb{R}$ are not homeomorphic.

10. Let $x = (x_1, x_2, \ldots) \in \mathbb{R}^\omega$. Let $\{x^{(n)}\}_{n \in \mathbb{Z}_+}$ be a sequence of points in $\mathbb{R}^\omega$ such that for each coordinate $i \in \mathbb{Z}_+$, the sequence $\{x_i^{(n)}\}_{n \in \mathbb{Z}_+}$ converges to x_i in $\mathbb{R}$. Show that $\{x^{(n)}\}_{n \in \mathbb{Z}_+}$ converges to x if $\mathbb{R}^\omega$ has the product topology.

11. Give an example of a topological space that is connected but not path-connected.

12. Define what it means for a topological space X to be a *Baire space*.

13. Show that if a topological space X is Hausdorff, then a sequence in X can converge to at most one point in X.

14. Show that the function $f\colon [0, 2\pi) \to \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ from the half-open interval to the circle defined by $f(t) = (\cos(t), \sin(t))$ is not a homeomorphism.

15. Let X be a locally compact Hausdorff space. Define the topology on $Y = X \cup \{\infty\}$ that makes Y the *one-point compactification of* X.