First Year Topology Exam, First Semester
January 2024

For the first five problems, show all your work and support all statements. Use a separate sheet of paper for each problem. (Each problem 10 points.)

1. Show there is no surjective function $f\colon X \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ is the power set of X.

2. Show that if X is a compact Hausdorff space, then X is a normal space.

3. Let (X, d) be a complete metric space. Let $f\colon X \to X$ and suppose there is some $c < 1$ with $d(f(x), f(y)) \leq c \cdot d(x, y)$ for all $x, y \in X$. Prove there exists a unique $x \in X$ with $f(x) = x$.

4. Let A be a proper subset of X, and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.

5. Let (X, d) be a metric space. Suppose there exists some $\varepsilon > 0$ such that for all $x \in X$, the closure $\overline{B_\varepsilon(x)}$ of the ε-ball about x is compact. Prove that X is complete.

Answer the following problems with complete definitions, complete statements, an example, or a short proof. (Each problem 5 points.)

6. Let $\{A_n\}_{n\in\mathbb{Z}_+}$ be a countable collection of countable sets. Prove that the union $\cup_{n\in\mathbb{Z}_+} A_n$ is countable.

7. Give an example of a topological space that is path-connected but not locally path-connected.

8. Let $\{X_\alpha\}_{\alpha\in A}$ be a collection of topological spaces. What is the product topology on $\prod_{\alpha\in A} X_\alpha$?

9. Show that $f\colon \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(x) = (x, x, x, \ldots)$ is not continuous if $\mathbb{R}^\omega$ has the box topology.

10. If $f\colon X \to Y$ is a continuous bijection with X compact and Y Hausdorff, then show that f^{-1} is continuous.

11. Give the definition of a *quotient map.*

12. Show that $[0, 1]$ and $(0, 1)$ are not homeomorphic.

13. Using the fact that there exists a continuous surjective function $[0, 1] \to [0, 1]^2$, show there is a continuous surjective function $[0, 1] \to [0, 1]^3$.

14. State the Baire Category Theorem.

15. Let X be a locally compact Hausdorff space. Define the topology on $Y = X \cup \{\infty\}$ that makes Y the *one-point compactification of* X.