1st Year 1st Semester Topology Exam
August, 2023

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Let $f : X \to X$ be a map of a compact metric space to itself that satisfies the following condition: $d(f(x), f(y)) < d(x, y)$.
(a) Prove that f is continuous.
(b) Show that f has a fixed point and the fixed point is unique.
2. Let $X \subset \mathbb{R}^2$ be a countable subset. Prove that $\mathbb{R}^2 \setminus X$ is connected.
3. Let X be a connected normal space having more than one point. Can X be countable?
4. Let A be a proper subset of X and let B be a proper subset of Y. If X and Y are connected, show that $(X \times Y) - (A \times B)$ is connected.
5. Let I_O denote the square $[0, 1] \times [0, 1]$ given the lexicographic order topology. Is I_O
(a) compact?
(b) connected?
(c) path connected?

Answer the following with complete definitions or statements or short proofs.

6. Give definition of Lebesgue number.
7. Show that the power set $\mathcal{P}(X)$ has strictly greater cardinality than X.
8. Show that the 2-sphere is not homeomorphic to
(a) the circle S^1;
(b) the 2-plane $\mathbb{R}^2$.
9. Show that every compact Hausdorff space is regular.
10. Give definition of a quotient map. Let

$$Z = \{(x, y) \in \mathbb{R}^2 \mid xy = 1\} \cup (\{0\} \times \mathbb{R}) \subset \mathbb{R} \times \mathbb{R}$$

be given the subspace topology. Is the projection $f : Z \to \mathbb{R}$ to the first coordinate, $f(x, y) = x$, a quotient map?