TOPOLOGY FIRST YEAR EXAM PART 1 • FALL 2022

Instructions: Do seven **(7)** and only seven of the following.

1. Prove that a compact Hausdorff space is normal.
2. Say $A \subset X$ is dense if $\overline{A} = X$.
   (a) Show that A is dense in X if and only if every nonempty open subset V in X satisfies $V \cap A \neq \emptyset$.
   (b) Assume that X and Y are topological spaces with Y Hausdorff and A is dense in X. Suppose that $f : X \to Y$ and $g : X \to Y$ are continuous functions with $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in X$.
3. Let $B \subset \mathbb{R}^{\mathbb{Z}_+}$ be defined as
$$B = \prod_{i \in \mathbb{Z}_+} (-1, 1) = (-1, 1) \times (-1, 1) \times (-1, 1) \times \dots$$
.
   (a) Is B open in the box topology?
   (b) Is B open in the uniform topology?
   (c) Is B open in the product topology?
4. If (X, d) is a metric space and $A \subset X$ for $x \in X$ define
$$d(x, A) = \inf\{d(x, a) \colon a \in A\}.$$
   (a) Show that the function $f : X \to [0, \infty)$ defined by $f(x) = d(x, A)$ is continuous
   (b) Show that $x \in \overline{A}$ if and only if $f(x) = 0$.
   (c) If A is compact then for any $x \in X$ there exists an $a \in A$ with $d(x, A) = d(x, a)$.
5. Show that (X, d) is complete if and only if any nested family of nonempty closed sets A_n with $\mathrm{diam}(A_n) \to 0$ has $\cap A_n \neq \emptyset$
6. Assume that $f : X \to Y$ is continuous and surjective.
   (a) If X is Lindelöf, show that Y is also.
   (b) If X is separable, show that Y is also.
7. If X and Y are connected topological spaces, show that $X \times Y$ is connected.
8. In the real line $\mathbb{R}$ let $\mathcal{T}$ be the collection of all subsets A of $\mathbb{R}$ such that $\mathbb{R} - A$ is either finite or all of $\mathbb{R}$.
   (a) Prove that $(\mathbb{R}, \mathcal{T})$ is topological space.
   (b) Is this topology Hausdorff?
   (c) To which point or points does the sequence $x_n = 1/n^2$ converge?