TOPOLOGY FIRST YEAR EXAM PART 1 • SPRING 2022

Instructions: Do seven **7** and only seven of the following.

1. Let X be a set, and let $\mathcal{T}$ be the collection of all subsets A of X such that $X - A$ is either finite or all of X.
   (a) Prove that $(X, \mathcal{T})$ is topological space.
   (b) Now let $X = \mathbb{R}$, the real line.
      (i) Determine if the sequence $x_n = 7n + 8$ converges and if so, to which point or points.
      (ii) Show that every infinite set is connected.
      (iii) What is $\overline{\{5\}}$?
2. Let X be a topological space. Let $D = \{(x, y) \in X \times X : x = y\}$. Prove that X is a Hausdorff space if and only if D is a closed subset of $X \times X$, where $X \times X$ has the product topology.
3. Show that $\mathbb{R}^{\mathbb{Z}_+}$ with the uniform topology is not connected.
4. A chain of subsets is a collection A_n for $n \in \mathbb{Z}_+$ with the property that for all i, $A_i \cap A_{i+1} \neq \emptyset$. If each A_n in a chain is connected, show that $\cup_{n \in \mathbb{Z}_+} A_n$ is also connected.
5. If A is a countable set, show that $\mathbb{R}^2 - A$ is path connected.
6. A space is called *locally Baire* if every point has an open neighborhood that is a Baire space. Prove that locally Baire implies Baire.
7. Show that the Tietze Extension Theorem implies Urysohn's Lemma.
8. Assume that (X, d) is metric. Show the following
   (a) X is normal
   (b) If X is compact then it is second countable