TOPOLOGY FIRST YEAR EXAM PART1 • FALL 2021

Instructions: Do **7** and only seven of the following.

1. Let $X \subset \mathbb{R}$ consist of all numbers of the form
$$1 + \frac{1}{\sqrt{p}} + \cdots + \frac{1}{\sqrt{p^n}}$$
with p prime and $n \in \mathbb{Z}_+$. Show that X is countable.

2. Assume $f : X \to Y$ is a continuous bijection with X compact and Y Hausdorff. Show that f is a homeomorphism.

3. Prove that a compact Hausdorff space is normal.

4. Assume that $f : X \to Y$ is continuous and surjective.
   (a) If X is Lindelöf, show that Y is also.
   (b) If X is separable, show that Y is also.

5. Each $A_\lambda \subset X$ is connected for $\lambda \in \Lambda$ and $\cap_{\lambda\in\Lambda} A_\lambda \neq \emptyset$ then prove that $\cup_{\lambda\in\Lambda} A_\lambda$ is connected.

6. Show that a subspace of a complete metric space is itself complete if and only if it is a closed subspace.

7. (a) Define the metric ρ on $[-1, 1]^{\mathbb{Z}_+}$ that gives it the uniform topology
   (b) Show that in this metric the unit ball, $B_1(\underline{0})$, is not limit point compact where $\underline{0}$ is the sequence $0, 0, 0, \ldots$.

8. Prove or disprove: The real line $\mathbb{R}_\ell$ with the lower limit topology is connected.