Numerical Linear Algebra Exam-May 2025
Do **4** (four) problems

1. Assume $A \in C^{m\times m}$.
(a) Show that A has a Schur decomposition.
(b) If A is normal, show that A is diagonalizable.

2. Suppose A is Hermitian positive definite.
(a) Prove that each principal submatrix of A is Hermitian positive definite.
(b) Prove that an element of A with largest magnitude lies on the diagonal.
(c) Prove that A has a Cholesky decomposition.

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$ for all $x \in R^n$. That is to find C and c such that $c\|x\|_\infty \le \|x\|_2 \le C\|x\|_\infty$, for all $x \in R^n$. Note that the constants should be determined so that the equalities hold for some nonzero $x \in R^n$.
(b) Show that $\|QA\|_2 = \|A\|_2$ if Q is a unitary matrix.

4. Assume that $A \in C^{n\times n}$ and there exists $p \ge 1$ such that $\|A\|_p < 1$, where $\|\cdot\|_p$ is a vector-induced matrix norm.
(a) Prove that $I - A$ is invertible.
(b) Prove that

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(c) Prove that $\|A\|_q\|A^{-1}\|_q \ge 1$, $\forall 1 \le q \le \infty$.
(d) Prove that

$$\frac{1}{1 + \|A\|_p} \le \|(I - A)^{-1}\|_p \le \frac{1}{1 - \|A\|_p}.$$

5. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in C^{m\times n}$. Let u_j denote column j of U.
(a) Suppose rank(A)=$p < n < m$. Show $\{u_1, u_2, \cdots, u_p\}$ is a basis for $Col(A)$ and $\{u_{p+1}, u_{p+2}, \cdots, u_m\}$ is a basis for $Null(A^*)$.

(b) Suppose A is full rank and $x \neq 0$. Let σ_i, $i = 1, \cdots, n$ be the singular values of A. Show

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that it holds.