Numerical Linear Algebra Exam-Jan 2025
Do **4** (four) problems

1. Assume $A \in C^{m\times m}$.
(a) Show that A has a Schur decomposition.
(b) If A has a collection of m linearly independent eigenvectors, show that A is diagonalizable.

2. If $q_1, \cdots, q_n$ is an orthonormal basis for the subspace $V \subset C^{m\times n}$ with $m > n$, prove that the orthogonal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.

3. Let $A \in C^{m\times n}$, with $m \geq n$ and $rank(A) = n \geq 3$. Let $a_1, a_2, \cdots$ denote the columns of A.
(a) Using the modified Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.
(b) Show the vector q_3 found in part (a) is orthogonal to both q_1 and q_2.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.
(a) If E is a $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1-\|E\|}.$$

(b) If A is a $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\|E\|}.$$

5. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in C^{m\times n}$. Let u_j denote column j of U.
(a) Suppose rank(A)=$p < n < m$. Show $\{u_1, u_2, \cdots, u_p\}$ is a basis for $Col(A)$ and $\{u_{p+1}, u_{p+2}, \cdots, u_m\}$ is a basis for $Null(A^*)$.
(b) Suppose A is full rank and $x \neq 0$. Let σ_i, $i = 1, \cdots, n$ be the sigular values of A. Show

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that it holds.