Numerical Linear Algebra, January, 2026.
Do **4** (four) problems.

1. Prove each of the following.

(a) The matrix 2-norm is invariant under unitary transformation.

(b) If $n \times n$ matrix A is diagonal, then for any matrix p-norm $\|\cdot\|_p$, it holds that $\|A\|_p = \max_{1 \le i \le n} |a_{ii}|$.

2. Suppose $A \in \mathbb{C}^{m \times n}$. Prove or provide a counterexample for each of the following.

(a) The Frobenius norm of a matrix is induced by (subordinate to) a vector norm.

(b) $\|A\|_2 \le \|A\|_F \le \sqrt{n}\|A\|_2$.

3. Let $A \in \mathbb{C}^{n \times n}$. You may use the fact (if you find it useful) That A has a Schur decomposition. Prove or provide a counterexample for each of the following.

(a) If A is both normal and triangular then it is diagonal.

(b) If A is normal then the 2-norm of A is equal to it's spectral radius $\rho(A)$.

4. **(a)** Let $r_i = \sum_{j=1, j \ne i}^{n} |a_{ij}|$. Let D_i be the disk in $\mathbb{C}$ with center a_{ii} and radius r_i. Prove that if λ is an eigenvalue of A, then $\lambda \in \bigcup_i D_i$; in other words, λ is in at least one of the disks D_i.

(b) Determine all eigenvalues of the Householder reflector $H(w)$, including their multiplicities. Justify your answer.

5. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 3 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find a full singular value decomposition of A.

(b) Find an economy QR decomposition of B.