Numerical Linear Algebra Exam-May 2024
Do **4** (four) problems

1. Let A be an $m \times n$ matrix ($m \geq n$) and let $A = \hat{Q}\hat{R}$ be a reduced QR factorization of A. Prove that A has full rank if and only if the diagonal entries of R are nonzero.

2. Assume that $A \in C^{n\times n}$ and there exists $p \geq 1$ such that $\|A\|_p < 1$, where $\|\cdot\|_p$ is a vector-induced matrix norm.
(a) Prove that $I - A$ is invertible.
(b) Prove that

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

(c) Prove that $\|A\|_q \|A^{-1}\|_q \geq 1$, $\forall 1 \leq q \leq \infty$.
(d) Prove that

$$\frac{1}{1 + \|A\|_p} \leq \|(I - A)^{-1}\|_p \leq \frac{1}{1 - \|A\|_p}.$$

3. Suppose A is Hermitian positive definite.
(a) Prove that each principal submatrix of A is Hermitian positive definite.
(b) Prove that an element of A with largest magnitude lies on the diagonal.
(c) Prove that A has a Cholesky decomposition.

4. Let $\mathbf{v} = [2, -1, 1]^T$ and $H = (span\{\mathbf{v}\})^{\perp}$.
(a) Find the matrix $P_{\mathbf{v}}$, the orthogonal projection onto $span\{\mathbf{v}\}$.
(b) Find the matrix P_H, the orthogonal projection onto H.
(c) Find Q_H, the unitary matrix that reflects across H.

5. Let A, $\delta A \in R^{n\times n}$ be full rank and b, x and $\delta x \in R^n$. Prove that if

$$Ax = b, \text{and } (A + \delta A)(x + \delta x) = b,$$

then

$$\frac{\|\delta A\|}{\|x + \delta x\|} \leq \kappa(A)\frac{\|\delta A\|}{\|A\|},$$

where $\kappa(A)$ is the condition number of A.